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Extension of Grimus–Stockinger formula from operator expansion of free Green function

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The operator expansion of free Green function of Helmholtz equation for arbitrary N-dimensional space leads to asymptotic extension of three dimensions Grimus–Stockinger formula closely related to multipole expansion. Analytical examples inspired by neutrino oscillation and neutrino deficit problems are considered for relevant class of wave packets.

Keywords: Asymptotic expansion; Green function; wave packet; neutrino deficit.

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1. Introduction

Considerable efforts were made recently[1–3] to extend the so-called Grimus–Stockinger theorem,[4] which is the main tool of the modern theory of neutrino oscillations[5–7] and gives the leading asymptotic behavior with $R = |\mathbf{R}| \to \infty$ for the integral:

$$\mathcal{J}(\mathbf{R}) = \int \frac{d^3q}{(2\pi)^3} \frac{e^{-i(\mathbf{q}\cdot\mathbf{R})}\Phi(\mathbf{q})}{(\mathbf{q}^2 - k^2 - i0)} \sim \frac{e^{ikR}}{4\pi R}\Phi(-k\mathbf{n})[1 + O(R^{-1/2})]\,, \tag{1}$$

where $\mathbf{R} = R\mathbf{n}$, $\mathbf{n}^2 = 1$, and the function $\Phi(\mathbf{q}) \in C^3$ decreases at least like $1/\mathbf{q}^2$ together with its first and second derivatives. In Sec. 2 it is reminded that the possibility of further asymptotic expansion and the order of leading correction depend[8] on the chosen properties[1,2] of this function. In Sec. 3, for appropriate space of $\Phi(\mathbf{q})$ a closed formula and simple recurrent relation for coefficients of asymptotic expansion of $\mathcal{J}(\mathbf{R})$ in all orders of R^{-s} are obtained. Such extension has particular importance for explanation[2,3] of observed[9] (anti-) neutrino deficit at short distances from the sources discussed in Sec. 4. But it may find much more wide implementation in quantum physics and optics, when the Green function of Helmholtz equation is used. In Secs. 4 and 5, the space chosen for the functions $\Phi(\mathbf{q})$ is advocated on the ground of quantum field theory of wave packets.[6,10] Some useful relations and generalization onto N-dimensional case are placed in the Appendices.

2. Preliminaries

In order to understand the physical nature of asymptotic expansion, we notice that for infinitely differentiable $\Phi(\mathbf{q})$ uniquely representable by its Taylor expansion for any finite $|\mathbf{q}| < \infty$: $e^{-i(\mathbf{q}\cdot\mathbf{R})}\Phi(\mathbf{q}) = \Phi(i\boldsymbol{\nabla}_{\mathbf{R}})e^{-i(\mathbf{q}\cdot\mathbf{R})}$, and then, formally:

$$\mathcal{J}(\mathbf{R}) = \Phi(i\boldsymbol{\nabla}_{\mathbf{R}})\frac{e^{ikR}}{4\pi R} = \Phi(-i\boldsymbol{\nabla}_{\mathbf{x}})\frac{e^{ik|\mathbf{R}-\mathbf{x}|}}{4\pi|\mathbf{R}-\mathbf{x}|}\bigg|_{\mathbf{x}=0}, \tag{2}$$

where the differential vector operator in spherical basis $\mathbf{n}, \boldsymbol{\eta}_\vartheta, \boldsymbol{\eta}_\varphi$ has the following properties:

$$\mathbf{n} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta), \qquad \boldsymbol{\eta}_\vartheta = \partial_\vartheta\mathbf{n}, \qquad \sin\vartheta\boldsymbol{\eta}_\varphi = \partial_\varphi\mathbf{n}, \tag{3}$$

$$\boldsymbol{\nabla}_{\mathbf{R}} = \mathbf{n}\partial_R + \frac{1}{R}\boldsymbol{\partial}_{\mathbf{n}}, \qquad (\mathbf{n}\cdot\boldsymbol{\nabla}_{\mathbf{R}}) = \partial_R, \qquad \boldsymbol{\partial}_{\mathbf{n}} = \boldsymbol{\eta}_\vartheta\partial_\vartheta + \frac{\boldsymbol{\eta}_\varphi}{\sin\vartheta}\partial_\varphi, \tag{4}$$

$$(\mathbf{n}\cdot\boldsymbol{\partial}_{\mathbf{n}}) = 0, \qquad (\boldsymbol{\partial}_{\mathbf{n}}\cdot\mathbf{n}) = 2, \qquad (\mathbf{n}\times\boldsymbol{\partial}_{\mathbf{n}})^2 = \boldsymbol{\partial}_{\mathbf{n}}^2, \qquad (\mathbf{n}\times\boldsymbol{\partial}_{\mathbf{n}}) = i\mathbf{L}_{\mathbf{n}}, \tag{5}$$

$$-\boldsymbol{\partial}_{\mathbf{n}}^2 = \mathbf{L}_{\mathbf{n}}^2 = 2R(\mathbf{n}\cdot\boldsymbol{\nabla}_{\mathbf{R}}) + R^2((\mathbf{n}\cdot\boldsymbol{\nabla}_{\mathbf{R}})^2 - \boldsymbol{\nabla}_{\mathbf{R}}^2), \quad \text{whence}, \tag{6}$$

$$\text{for} \quad \cos\vartheta = c: \ \mathbf{L}_{\mathbf{n}}^2 = -[\partial_c(1-c^2)\partial_c + (1-c^2)^{-1}\partial_\varphi^2] \equiv \mathcal{L}_{\mathbf{n}}, \tag{7}$$

and the well-known representation at point $\mathbf{R}$ for the spherical wave coming from point $\mathbf{x}$, as a free Schrödinger's three-dimensional Green function,[11] is used (see (A.6)):

$$\frac{e^{ik|\mathbf{R}-\mathbf{x}|}}{4\pi|\mathbf{R}-\mathbf{x}|} = \int\frac{d^3q}{(2\pi)^3}\frac{e^{\pm i(\mathbf{q}\cdot(\mathbf{R}-\mathbf{x}))}}{(\mathbf{q}^2-k^2-i0)}, \tag{8}$$

which for $\mathbf{x} = 0$ satisfies:

$$(-\boldsymbol{\nabla}_{\mathbf{R}}^2 - k^2)\frac{e^{\pm ikR}}{4\pi R} = \delta_3(\mathbf{R}), \tag{9}$$

where for $R > 0$ $\boldsymbol{\nabla}_{\mathbf{R}}^2$ is given also by (A.1) with $N = 3$. Since for $R > 0$ the right hand side of this equation is zero, $\delta_3(\mathbf{R}) = 0$, for spherically symmetric case one immediately obtains from (2) formally an exact answer, which really takes place at least for the function $\Psi(k^2)$ regular and bounded in upper half plane $\operatorname{Im} k \geq 0$ of complex variable k:

$$\text{when} \ \ \Phi(\mathbf{q}) = \Psi(q^2), \quad \text{for} \ \ \mathbf{q}^2 = q^2, \ \ \text{then} \ \ \mathcal{J}(\mathbf{R}) = \Psi(k^2)e^{ikR}/(4\pi R). \tag{10}$$

Therefore, the higher order corrections originate only by asymmetry of function $\Phi(\mathbf{q})$ relative to the directions of $\mathbf{q}$ in accordance with our previous result.[1] Indeed, in order to obtain them, we supposed[1] that $\Phi(\mathbf{q})$ and its first and second derivatives are represented by Fourier transforms as:

$$\Phi(\mathbf{q}) = \int d^3x\, e^{i(\mathbf{q}\cdot\mathbf{x})}\phi(\mathbf{x})\,, \qquad \boldsymbol{\nabla}_q\Phi(\mathbf{q}) = i\int d^3x\, e^{i(\mathbf{q}\cdot\mathbf{x})}\mathbf{x}\phi(\mathbf{x}) \tag{11}$$

and so on. Then by interchanging the order of integration and using Eq. (8) we found the representation:

$$\mathcal{J}(\mathbf{R}) = \int d^3x \frac{e^{ik|\mathbf{R}-\mathbf{x}|}}{4\pi|\mathbf{R}-\mathbf{x}|}\phi(\mathbf{x})\,. \tag{12}$$

Substituting here the expansion, which in the exponential should always contain one additional order in comparison with the number of orders in denominator, for:

$$|\mathbf{R}-\mathbf{x}| = R\left[1 - 2\frac{(\mathbf{n}\cdot\mathbf{x})}{R} + \frac{\mathbf{x}^2}{R^2}\right]^{1/2}, \text{ with } \mathbf{x}^2 - (\mathbf{n}\cdot\mathbf{x})^2 \risingdotseq (\mathbf{n}\times\mathbf{x})^2\,, \tag{13}$$

$$|\mathbf{R}-\mathbf{x}| = R - (\mathbf{n}\cdot\mathbf{x}) + \frac{\mathbf{x}^2 - (\mathbf{n}\cdot\mathbf{x})^2}{2R} + \cdots\,, \tag{14}$$

we had the corresponding expansion of integral (12) up to $O(R^{-3})$:

$$\mathcal{J}(\mathbf{R}) = \frac{e^{ikR}}{4\pi R}\int d^3x\, e^{-ik(\mathbf{n}\cdot\mathbf{x})}\phi(\mathbf{x})\left[1 + \frac{(\mathbf{n}\cdot\mathbf{x})}{R} + \frac{ik}{2R}(\mathbf{x}^2 - (\mathbf{n}\cdot\mathbf{x})^2) + \cdots\right],$$

that by making use of (11) was immediately transcribed as:

$$\mathcal{J}(\mathbf{R}) = \frac{e^{ikR}}{4\pi R}\left[1 - \frac{i}{R}(\mathbf{n}\cdot\boldsymbol{\nabla}_q) + \frac{ik}{2R}((\mathbf{n}\cdot\boldsymbol{\nabla}_q)^2 - \boldsymbol{\nabla}_q^2) + \cdots\right]\Phi(\mathbf{q})\bigg|_{\mathbf{q}=-k\mathbf{n}}\,, \tag{15}$$

with

$$(\mathbf{n}\cdot\boldsymbol{\nabla}_q)\Phi(\mathbf{q})\big|_{\mathbf{q}=-k\mathbf{n}} = -(\mathbf{n}\cdot\boldsymbol{\nabla}_{\mathbf{k}})\Phi(-\mathbf{k}) = -\partial_k\Phi(-k\mathbf{n}) \tag{16}$$

and so on. The next observation is that due to (16), with $\mathbf{k} = k\mathbf{n}$, for $\mathbf{q} = -\mathbf{k}$, $\boldsymbol{\nabla}_{\mathbf{q}} = -\boldsymbol{\nabla}_{\mathbf{k}}$, from (6) and (7) with $\mathbf{R} \mapsto \mathbf{k}$, one has: $2k(\mathbf{n}\cdot\boldsymbol{\nabla}_{\mathbf{k}}) + k^2((\mathbf{n}\cdot\boldsymbol{\nabla}_{\mathbf{k}})^2 - \boldsymbol{\nabla}_{\mathbf{k}}^2) = \mathcal{L}_{\mathbf{n}}$, and expression (15) takes the following simple form:

$$\mathcal{J}(\mathbf{R}) = \frac{e^{ikR}}{4\pi R}\left[1 + \frac{i}{R}(\mathbf{n}\cdot\boldsymbol{\nabla}_{\mathbf{k}}) + \frac{ik}{2R}((\mathbf{n}\cdot\boldsymbol{\nabla}_{\mathbf{k}})^2 - \boldsymbol{\nabla}_{\mathbf{k}}^2) + \cdots\right]\Phi(-\mathbf{k}) \tag{17}$$

$$= \frac{e^{ikR}}{4\pi R}\left[1 + \frac{i}{2kR}\mathcal{L}_{\mathbf{n}} + O(R^{-2})\right]\Phi(-k\mathbf{n})\,. \tag{18}$$

In the next section by using the another surprisingly simple way, this result is generalized onto all orders of R^{-s}. It should be noted that in spite of the one and the same final results of both Eqs. (15), (17) and (18), the substitution like (13): $\boldsymbol{\nabla}_q^2 - (\mathbf{n}\cdot\boldsymbol{\nabla}_q)^2 \leftrightharpoons (\mathbf{n}\times\boldsymbol{\nabla}_q)^2$ is correct till $\mathbf{q} \neq -k\mathbf{n}$ for expressions[1–3] like (15), but it becomes incorrect for noncommutative operators $\mathbf{n}$ and $\boldsymbol{\nabla}_{\mathbf{k}}$ in expressions like (17). This difference from Refs. 1–3 below eventually gives rise to an explicit operator expression for higher coefficients of this asymptotic expansion with arbitrary s.

3. Asymptotic and Multipole Expansions

Lemma 1. *For* $\mathbf{R} = R\mathbf{n}$, $\mathbf{x} = r\mathbf{v}$, $\mathbf{v} = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta)$, $|\mathbf{x}| = r < R$, *with operator* $\mathcal{L}_{\mathbf{n}} = \mathbf{L}_{\mathbf{n}}^2$ *(or* $\mathbf{n} \mapsto \mathbf{v}$*) defined by Eqs.* (5)–(7) *and positively defined operator* $\mathcal{L}_{\mathbf{n}} + \frac{1}{4} = (\Lambda_{\mathbf{n}} + \frac{1}{2})^2$, *so that* $\Lambda_{\mathbf{n}} + \frac{1}{2} = \sqrt{\mathcal{L}_{\mathbf{n}} + \frac{1}{4}}$ *is positively defined*:

$$\frac{e^{ik|\mathbf{R}-\mathbf{x}|}}{4\pi|\mathbf{R}-\mathbf{x}|} = \frac{\chi_{\Lambda_{\mathbf{n}}}(-ikR)}{4\pi R} e^{-ik(\mathbf{n}\cdot\mathbf{x})} \tag{19}$$

$$\sim \frac{e^{ikR}}{4\pi R}\left\{1 + \sum_{s=1}^{\infty} \frac{\prod_{\mu=1}^{s}[\mathcal{L}_{\mathbf{n}} - \mu(\mu-1)]}{s!(-2ikR)^s}\right\} e^{-ik(\mathbf{n}\cdot\mathbf{x})}\,. \tag{20}$$

Proof. The expression (19) is a formal operator rewritten for $R > r$ of the usual multipole expansion of the Green function[11] (8) via the corresponding expansion of the plane wave,[11] given also by formulas (8.533) and (8.534) of Ref. 12, with the formally introduced instead l, but not really appeared operator $l \mapsto \Lambda_{\mathbf{n}}$:

$$\frac{e^{\pm ik|\mathbf{R}-\mathbf{x}|}}{4\pi|\mathbf{R}-\mathbf{x}|} = \frac{1}{kRr}\sum_{l=0}^{\infty} i^{\mp l}\chi_l(\mp ikR)\psi_{l\,0}(kr)\sum_{m=-l}^{l} Y_l^m(\mathbf{n})\overset{*}{Y}{}_l^m(\mathbf{v})\,, \tag{21}$$

$$e^{\mp i(\mathbf{k}\cdot\mathbf{x})} = \frac{4\pi}{kr}\sum_{l=0}^{\infty} i^{\mp l}\,\psi_{l\,0}(kr)\sum_{m=-l}^{l} Y_l^m(\mathbf{n})\overset{*}{Y}{}_l^m(\mathbf{v})\,. \tag{22}$$

Here the spherical functions $Y_l^m(\mathbf{n}) = \langle\mathbf{n}|l\,m\rangle$ and Legendre polynomials $P_l(\xi)$ for $\xi = c = \cos\vartheta$ or $\xi = (\mathbf{n}\cdot\mathbf{v})$, as eigenfunctions of self-adjoint operator (6) and (7) on the unit sphere, satisfy the well-known orthogonality, parity and completeness conditions[11–14] with delta-function $\delta_\Omega(\mathbf{n},\mathbf{v})$ on the unit sphere:

$$\mathbf{L}_{\mathbf{n}}^2 Y_l^m(\mathbf{n}) = l(l+1)Y_l^m(\mathbf{n})\,, \qquad \mathbf{L}_{\mathbf{n}}^2 P_l(\xi) = l(l+1)P_l(\xi)\,, \tag{23}$$

$$\int d\Omega(\mathbf{n})\overset{*}{Y}{}_l^m(\mathbf{n})Y_j^{m'}(\mathbf{n}) = \delta_{lj}\delta_{mm'}\,, \qquad \sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi}P_l((\mathbf{n}\cdot\mathbf{v})) = \delta_\Omega(\mathbf{n},\mathbf{v})\,, \tag{24}$$

$$\sum_{m=-l}^{l} Y_l^m(\mathbf{n})\overset{*}{Y}{}_l^m(\mathbf{v}) \equiv \frac{(2l+1)}{4\pi}P_l((\mathbf{n}\cdot\mathbf{v}))\,, \qquad (-1)^l Y_l^m(\mathbf{n}) = Y_l^m(-\mathbf{n})\,. \tag{25}$$

The solutions $\chi_l(\mp ikr)$, $\psi_{l\,0}(kr)$ of free radial Schrödinger equation:

$$\left[r^2\left(\frac{1}{r}\partial_r^2 r + k^2\right)\right]\frac{\psi_{l\,0}(kr)}{r} = l(l+1)\frac{\psi_{l\,0}(kr)}{r}\,, \tag{26}$$

are defined by Macdonald $K_\lambda(z)$ and Bessel $J_\lambda(y)$ functions,[8,11–14] that for integer l, i.e. half integer $\lambda = l + \frac{1}{2}$ are reduced to elementary functions:[8,12]

$$\begin{aligned} \chi_l(bR) &\equiv \left(\frac{2bR}{\pi}\right)^{1/2} K_{l+\frac{1}{2}}(bR)\,, \\ \chi_l(bR)\Big|_{l=\mathrm{int}} &= e^{-bR}\sum_{s=0}^{l}\frac{(l+s)!}{s!(l-s)!(2bR)^s}\,, \end{aligned} \tag{27}$$

$$\psi_{l\,0}(kr) \equiv \left(\frac{\pi kr}{2}\right)^{1/2} J_{l+\frac{1}{2}}(kr) \equiv \frac{1}{2i}\left[i^{-l}\chi_l(-ikr) - i^l\chi_l(ikr)\right]. \tag{28}$$

The function $K_\lambda(z)$ (A.3) is a whole function[8,12] of λ^2, that is the reason, why the well-defined operator $\Lambda_{\mathbf{n}}$ introduced in (19) does not appear explicitly.

The expression (20) is the known asymptotic series of expression (19) at $R \to \infty$, as an infinite asymptotic version[8,12] of the sum (27) for arbitrary non-integer l, $|\arg(bR)| < 3\pi/2$. It directly results also from the substitution of finite sums (25) and (27) by interchanging the order of summations, that converts Eq. (21) into the sum:

$$\frac{e^{ikR}}{4\pi R}\sum_{s=0}^{\infty}\frac{1}{s!(-2ikR)^s}\,\frac{1}{kr}\sum_{l=s}^{\infty} i^{-l}\frac{(l+s)!}{(l-s)!}\,\psi_{l0}(kr)(2l+1)P_l((\mathbf{n}\cdot\mathbf{v}))\,, \tag{29}$$

where

$$\frac{(l+s)!}{(l-s)!} = \prod_{\mu=1}^{s}(l-\mu+1)(l+\mu) = \prod_{\mu=1}^{s}[l(l+1)-\mu(\mu-1)]\,, \tag{30}$$

equals to zero for all missing summands with $0 \le l \le s-1$ automatically and due to Eq. (23) may be factored out from the sum over l as operator product in the right-hand side of Eq. (20). The formal addition of all such missing in fact zero summands with $0 \le l \le s-1$ completes then sum over l into the plane wave (22) and converts the expression (29) into the expansion (20).

Remark. The operator $\mathcal{L}_{\mathbf{n}}$ in Eq. (20) with the same success may be replaced by operator in square brackets of the left-hand side of Eq. (26) or by the same with interchanging $r \rightleftharpoons k$. □

Theorem 1. *Let* $\Phi(\mathbf{q}) \in S(\mathbf{R}^3_{\mathbf{q}})$, *the space of functions infinitely differentiable* $\forall\,\mathbf{q} \in \mathbf{R}^3_{\mathbf{q}}$, *decrease faster than any power of* $1/|\mathbf{q}|$ *together with all its derivatives.*

Then integral $\mathcal{J}(\mathbf{R})$ (1), (12) at $R \to \infty$ admits asymptotic expansion, which has asymptotic sense[8] *even though $\phi(\mathbf{x})$ in Eq. (11) has a finite support*:

$$\mathcal{J}(\mathbf{R}) \sim \frac{e^{ikR}}{4\pi R}\Phi(-k\mathbf{n})\left\{1 + \sum_{s=1}^{\infty} \frac{C_s(k,\mathbf{n})}{(-2ikR)^s}\right\}, \quad \textit{with} \tag{31}$$

$$\Phi(-k\mathbf{n})C_s(k,\mathbf{n}) = \frac{1}{s!}\prod_{\mu=1}^{s}[\mathcal{L}_{\mathbf{n}} - \mu(\mu-1)]\Phi(-k\mathbf{n})\,, \quad \textit{or} \tag{32}$$

$$\Phi(-k\mathbf{n})C_s(k,\mathbf{n}) = \frac{\mathcal{L}_{\mathbf{n}} - s(s-1)}{s}\Phi(-k\mathbf{n})C_{s-1}(k,\mathbf{n})\,, \qquad C_0(k,\mathbf{n}) = 1\,, \tag{33}$$

and which is equivalent to infinite reordering of its multipole expansion:

$$\mathcal{J}(\mathbf{R}) \sim \frac{1}{4\pi R}\sum_{j=0}^{\infty} \chi_j(-ikR) \sum_{m=-j}^{j} B_j^m(k) Y_j^m(\mathbf{n})\,, \tag{34}$$

$$\textit{with} \qquad \Phi(-k\mathbf{n})C_s(k,\mathbf{n}) = \frac{1}{s!}\sum_{j=s}^{\infty}\frac{(j+s)!}{(j-s)!}\sum_{m=-j}^{j} B_j^m(k)Y_j^m(\mathbf{n})\,, \tag{35}$$

$$\textit{for} \qquad \Phi(-k\mathbf{n}) = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} B_j^m(k)Y_j^m(\mathbf{n})\,, \qquad \phi(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3}\Phi(\mathbf{q})e^{-i(\mathbf{q}\cdot\mathbf{x})}\,. \tag{36}$$

Proof. Since Fourier transformation (11) maps the space $S(\mathbf{R}^3)$ into itself[15] the function $\phi(\mathbf{x}) \in S(\mathbf{R}^3_{\mathbf{x}})$ also and is represented by the inverse Fourier transform (36). Let us suppose at first that $\phi(\mathbf{x})$ has a finite support at $|\mathbf{x}| \leq r_0$. Then for $R > r_0$ we can directly substitute the expressions (20) into representation (12) of $\mathcal{J}(\mathbf{R})$ with the following result after interchange of the order of summation, differentiation and integration of Fourier transform (11), justified[8] also for the asymptotic series:

$$\begin{aligned}\mathcal{J}(\mathbf{R}) &= \frac{\chi_{\Lambda_{\mathbf{n}}}(-ikR)}{4\pi R}\Phi(-k\mathbf{n}) \\ &\sim \frac{e^{ikR}}{4\pi R}\left\{1 + \sum_{s=1}^{\infty}\frac{\prod_{\mu=1}^{s}[\mathcal{L}_{\mathbf{n}} - \mu(\mu-1)]}{s!(-2ikR)^s}\right\}\Phi(-k\mathbf{n})\,.\end{aligned} \tag{37}$$

This is exactly asymptotic expansion (31) with dimensionless coefficients $C_s(k,\mathbf{n})$, defined by Eq. (32). However, that is not the case for the function $\phi(\mathbf{x})$ with infinite support. Estimating it for $r > R$ as $|\phi(\mathbf{x})| < C_M/r^M$ with arbitrary finite $M \gg 1$, the two pieces of corrections that should be added, are easy estimating as:

$$\begin{aligned}\Delta_R\mathcal{J} &= \int_{r>R} d^3x \frac{e^{ik|\mathbf{R}-\mathbf{x}|}}{4\pi|\mathbf{R}-\mathbf{x}|}\phi(\mathbf{x})\,, \\ \Delta_R\Phi &= -\frac{\chi_{\Lambda_{\mathbf{n}}}(-ikR)}{4\pi R}\int_{r>R} d^3x\, e^{-ik(\mathbf{n}\cdot\mathbf{x})}\phi(\mathbf{x})\,,\end{aligned} \tag{38}$$

$$|\Delta_R \mathcal{J}| < \frac{C_M}{(M-2)R^{M-2}}\,, \qquad |\Delta_R \Phi| < \frac{C_M}{(M-3)R^{M-2}}[1+O(R^{-1})]\,. \tag{39}$$

Due to these corrections and the arbitrariness of $M \gg 1$, the expansion (31) and (37) acquires an additional asymptotic meaning[8] in comparison with expansion (20). The terms with $s = 0, 1$ reproduce the results[1] (10) and (18) of previous section.

Coefficients of multipole expansion in (36) may be defined in the following two different ways, whose equivalence, for $\mathbf{x} = r\mathbf{v}$:

$$B_l^m(k) \equiv \int d\Omega(\mathbf{n})\Phi(-k\mathbf{n})\overset{*}{Y}{}_l^m(\mathbf{n}) = 4\pi \int d^3x \frac{\psi_{l0}(kr)}{kr} i^{-l} \overset{*}{Y}{}_l^m(\mathbf{v})\phi(\mathbf{x})\,, \tag{40}$$

follows from Eqs. (11), (22)–(25) and (A.4), with inverse Fourier transform (36).

Substitution of multipole expansion of Green function (21) into the integral (12) by making use of definition (40), after the same steps and under the same conditions for $\phi(\mathbf{x})$ as above leads to the **multipole** expansion (34), which with the help of expansion (27) transcribes again as asymptotic expansion (31) with the coefficients given by Eq. (35). But the same expression (35) is obtained by direct substitution of (36) into the definition (32) by means of (23) and (30). This confirms the equivalence of expansions (31) and (34).

Corollary 1. *From* (31) *with* $C_s = C_s(k, \mathbf{n})$, *for the squared absolute value follows*:

$$(4\pi R)^2 |\mathcal{J}(\mathbf{R})|^2 \sim |\Phi(-k\mathbf{n})|^2 \left\{ 1 + \sum_{s=1}^{\infty} \frac{i^s [C_s + (-1)^s \overset{*}{C}_s]}{(2kR)^s} \right.$$
$$\left. + \sum_{\zeta=2}^{\infty} \frac{i^\zeta}{(2kR)^\zeta} \sum_{s=1}^{\zeta-1} (-1)^{\zeta-s} C_s \overset{*}{C}_{\zeta-s} \right\}. \tag{41}$$

For $\Phi(-\mathbf{k}) = \overset{*}{\Phi}(-\mathbf{k})$: $C_s = \overset{*}{C}_s$, *so* $s \mapsto 2n$ *at the first sum over* s. *But the second internal sum over* $1 \le s \le \zeta - 1$ *in* (41) *is equal to itself with the multiplier* $(-1)^\zeta$ *and thus* $\zeta \mapsto 2n$ *also. Dividing further this sum over* $1 \le s \le 2n-1$ *into two parts*: *with* $1 \le s \le n$ *and* $n+1 \le s \le 2n-1$, *and putting for the second sum* $s = 2n - s'$, *one finds*:

$$(4\pi R)^2 |\mathcal{J}(\mathbf{R})|^2 \sim \Phi^2(-k\mathbf{n}) \left\{ 1 + \sum_{n=1}^{\infty} \frac{\Upsilon_n}{(2kR)^{2n}} \right\}, \quad \textit{with} \quad \Upsilon_n = \Upsilon_n(k, \mathbf{n})\,, \tag{42}$$

$$\Upsilon_n = (-1)^n \left[2C_{2n} + \sum_{s=1}^{2n-1} (-1)^s C_s C_{2n-s} \right] = (-1)^n \sum_{s=0}^{2n} (-1)^s C_s C_{2n-s}\,, \tag{43}$$

$$\textit{or} \quad \Upsilon_n = 2(-1)^n \sum_{s=0}^{n} (-1)^s C_s C_{2n-s} - (C_n)^2\,, \qquad \Upsilon_1 = (C_1)^2 - 2C_2\,, \tag{44}$$

which, due to "homogeneity" over C_s relative to index s "like a power", coincides with the relations[2] *for dimensional analogs of coefficients C_s and Υ_s.*

Corollary 2. *Asymptotic expansion of the integral $\tilde{\mathcal{J}}(\mathbf{R}) = \mathcal{J}(-\mathbf{R})$* (1) *is obtained by substitutions $\mathbf{n} \mapsto -\mathbf{n}$, $\mathcal{L}_{-\mathbf{n}} = \mathcal{L}_{\mathbf{n}}$* (7) *in Eqs.* (31)–(33), (37) *and* (41)–(44). □

The explicit Eqs. (31) and (32) and recurrent relation (33) may be compared with seminumerical calculations[2,3] of the same expansion, that for arbitrary $\Phi(\mathbf{q}) \in S(\mathbf{R}^3_{\mathbf{q}})$ have to give the same coefficients for the same orders R^{-s}. For the first correction this is easily seen from Eqs. (15)–(18) and Eq. (8) of Ref. 2 respectively with substitution $\boldsymbol{\nabla}^2_q - (\mathbf{n} \cdot \boldsymbol{\nabla}_q)^2 \leftrightharpoons (\mathbf{n} \times \boldsymbol{\nabla}_q)^2$. The same takes place for the next few corrections computed there. It is clear that any common multiplicative dependence of function $\Phi(\mathbf{q})$ on $|\mathbf{q}| = k$ does not affect the coefficients $C_s(k, \mathbf{n})$ (32) and $\Upsilon_n(k, \mathbf{n})$ (43). But there are no ways to trace this property for arbitrary order s with only numerically calculable coefficients in Eqs. (41)–(46) of Ref. 2. An important advantage of the results (31)–(33) for coefficients of asymptotic expansion is not only the explicit arbitrariness of their order s, but to elucidate, that in fact they are defined by dependence of $\Phi(\mp k\mathbf{n})$ on the *unit vector* $\mathbf{n}$ *only*. This essentially reduces the number of possible degrees of freedom and simplifies further analytical consideration. Indeed, when $B^m_l(k) = 0$ for $l > j$, the dependence (33), (35) and (44) of Υ_s on function $\Phi(\mathbf{q})$ (36) leads to termination of the asymptotic expansion: $C_s = 0 = \Upsilon_s$, for $s > j$, which is also clear from Eq. (34). If one has:

$$\Phi(-k\mathbf{n}) = \sum_{m=-j}^{j} B^m_j(k) Y^m_j(\mathbf{n}) , \quad \text{then}$$
$$C_n = \frac{(j+n)!}{n!(j-n)!} , \qquad \Upsilon_1 = 2j(j+1) > 0 , \tag{45}$$

$$\Upsilon_j = (C_j)^2 , \qquad \Upsilon_n = \frac{2(-1)^n}{(2n)!} \sum_{s=0}^{n} (-1)^s C^s_{2n} \frac{(j+s)!(j+2n-s)!}{(j-s)!(j-2n+s)!} - (C_n)^2 , \tag{46}$$

where $0 \leq n \leq j$, $\max(0, 2n-j) \leq s \leq n$ and C^s_{2n} are binomial coefficients.

4. Neutrino Deficit and Wave Packets

At diagrammatic treatment of neutrino oscillation[2–7] the function $\Phi(\mathbf{q})$ in Eq. (1) represents an overlap function $\mathcal{F}(q) = \mathcal{F}_C(q)\mathcal{F}_D(q)$ defined for $q^\mu = (q^0, \mathbf{q})$ as a real-valued product of convolutions of the wave packets of external particles participating in the processes of (anti-) neutrino creation $\{C\}$ and detection $\{D\}$ at respective spacetime points. According to the Feynman rules for respective tree amplitude[4–7] one has to deal with asymptotic expansion of $\mathcal{J}(\mathbf{R})$ (1) for the case of antineutrino[4] or with expansion of $\tilde{\mathcal{J}}(\mathbf{R})$ for the case of neutrino,[5] where the vector $\mathbf{R} = R\mathbf{n} = \mathbf{X}_D - \mathbf{X}_C$ for both cases defines the macroscopic geometrical parameters of observation:[5] distance R and direction $\mathbf{n}$. The event rate is obtained to be proportional to $|\mathcal{J}(\mathbf{R})|^2$ or $|\tilde{\mathcal{J}}(\mathbf{R})|^2$ respectively (see for detail Refs. 2 and 5–7), that due to (42) in the leading and next-to-leading orders reads:

$$\propto \frac{\Phi^2(\mp k\mathbf{n})}{R^2}\left\{1+\frac{\Upsilon_1(k,\pm\mathbf{n})}{(2kR)^2}+\cdots\right\}. \tag{47}$$

Here the absolute value of momentum $k=\sqrt{q_0^2-m_j^2}$ is defined by neutrino mass m_j and its mean energy $q^0=\mp Q^0$ near its most probable energies $Q_C^0 \approx Q_D^0$, defined in turn[4–7] by fixed parameters of respective external wave packets, such as their masses m_a, four-vectors of most probable energy–momentum p_a and corresponding widths $\sigma_a \ll m_a$, because in the plane wave limit, $\sigma_a \to 0$, $\forall\, a$ the overlap function keeps exact energy–momentum conservation for both vertices, fully destroying the neutrino-oscillations pattern,[5] $\mathcal{F}(q)|_{\sigma_a\to 0} \propto \delta_4(q \pm Q_C)\delta_4(q \pm Q_D)$, where Q_C becomes then a full four-momentum incoming into (anti-) neutrino propagator, whereas Q_D is full with its outgoing four-momentum. For $\sigma_a > 0$ the function $\mathcal{F}(q)$ can provide thus only approximate or "smeared" conservation[6] of some mean (anti-) neutrino energy $Q^0 \approx Q_C^0 \approx Q_D^0$ and some mean (anti-) neutrino three-momentum $\mathbf{Q} \approx \mathbf{Q}_C \approx \mathbf{Q}_D$, whose explicit form depends on the chosen models of wave packets and the used approximations.[5–7] So, the function $\Phi(\mp\, k\mathbf{n}) = \mathcal{F}(q^0,\mathbf{q})|_{\mathbf{q}=\mp k\mathbf{n}}$ may be more or less sharply peaked[5,6] and varies rapidly: near the point $k\mathbf{n} \simeq \mathbf{Q} = |\mathbf{Q}|\boldsymbol{\rho}$, for the more rough case[5] (a); or near the two adjacent points $k\mathbf{n} \simeq \mathbf{Q}_{C,D} = |\mathbf{Q}_{C,D}|\boldsymbol{\rho}_{C,D}$, where $\boldsymbol{\rho}^2_{C,D} = 1$, for the more precisely case[6] (b). For the case (a), up to fully unessential now multiplicative dependence on k we remain with a function $\Phi(\mp\, k\mathbf{n}) \mapsto f(\xi)$ of the one dimensionless variable $\xi = (\boldsymbol{\rho}\cdot\mathbf{n})$ only. For the case (b) without full symmetry relative to vectors $\boldsymbol{\rho}_{C,D}$, we remain with a function $\Phi(\mp\, k\mathbf{n}) \mapsto \mathcal{W}(\xi_C,\xi_D)$ of two independent variables $\xi_{C,D} = (\boldsymbol{\rho}_{C,D}\cdot\mathbf{n})$, which in Gassian case[5,7] or in sharply peak approximation[6] always contains a function $\Phi(\mp\, k\mathbf{n}) \propto \mathcal{H}(\zeta)$ of the one dimensionless variable $\zeta = (\mathbf{nBn}) > 0$ with the dimensionless symmetrical positively defined tensor $\mathbf{B}$ built on the components of these vectors, whose traceless part $\mathbf{B}_0$ only is in fact necessary. Some techniques with these functions are collected in Appendix B.

When $\Upsilon_1(k,\pm\mathbf{n}) < 0$, the last multiplier in expression (47) leads to suppression factor to the usual inverse square law for the event rate that reads as:

$$\propto \frac{1}{R^2}\left\{1-\frac{\varrho_0^2}{R^2}\right\}, \quad \text{where} \quad \varrho_0^2 = -\frac{\bar{\Upsilon}_1(k,\pm\mathbf{n})}{(2k)^2}\,, \tag{48}$$

and the averaging $\bar{\Upsilon}_1$ relative to directions of $\boldsymbol{\rho}$ is implied for the case (a), that in fact approximately puts $\boldsymbol{\rho} = \mathbf{n}$, $\xi = 1$. For the case (b), one has to average relative to the vectors $\boldsymbol{\rho}_{C,D}$ independently, what makes the final result more model-dependent. According to Refs. 2 and 3 this suppression factor provided by the higher order corrections to Grimus–Stockinger formula (1) naturally explains the observed[9] (anti-) neutrino deficit in reactor and others short-baseline experiments (see Refs. 2, 3 and 9 and references therein). Such explanation does not require any "new physics" and perhaps[2] can be tested experimentally.

Equations (32) and (44) show that negative sign of coefficient Υ_1 originates by the most rapid variation $\Phi(\mp k\mathbf{n})$ with vector $\mathbf{n}$. Exponential variations are directly generated by the above limiting properties of the overlap function and are naturally arising in popular models of wave packets.[5–7] Since beyond the oscillation problem it is not meaningless to leave overlap function be defined by one wave packet only, any model of wave packet also has to reproduce the above limiting properties of $\mathcal{F}_{C,D}(q)$.

In the spirit of Ref. 6, the following form of wave packet in momentum representation was suggested[10] for $q_\mu\zeta_a^\mu \equiv (q\zeta_a) > 0$, with time-like four-vector ζ_a^μ of mean quantum numbers of wave packet, such that $\zeta_a^0 > 0$, $\zeta_{a\mu}\zeta_a^\mu \equiv \zeta_a^2 > 0$:

$$\begin{aligned} \zeta_a(p_a,\sigma_a) &= p_a g_1(m_a,\sigma_a) + s_a g_2(m_a,\sigma_a)\,,\\ \phi^\sigma(\mathbf{q},\mathbf{p}_a) &= N_\sigma(m_a,\zeta_a^2)e^{-(q\zeta_a)}\,, \end{aligned} \tag{49}$$

where both the momentum four-vectors $q^\mu = (E_q,\mathbf{q})$, $p_a^\mu = (E_{p_a},\mathbf{p}_a)$ are on mass shell: $E_q = \sqrt{\mathbf{q}^2+m_a^2} > 0$, and so on, s_a is a spin four-vector, $g_1 \gg |g_2|$ are some real functions[10] of mass m_a and width σ_a, and $\aleph(\tau)m_a^{-2} = N_\sigma > 0$, is normalization constant with fixed asymptotic behavior[10] as a function of dimensionless invariant variable $\tau = m_a\sqrt{\zeta_a^2(p_a,\sigma_a)}$, at $\tau\to\infty$ ($\sigma_a \to 0$), and at $\tau \to 0$ ($\sigma_a\to\infty$):

$$\aleph(\tau) \underset{\tau\to\infty}{\longmapsto} 2(2\pi)^{3/2}\tau^{3/2}e^\tau \quad \text{and} \quad \aleph(\tau) \underset{\tau\to 0}{\longmapsto} \aleph(0) > 0\,. \tag{50}$$

Coordinate representation of this wave packet with fixed center x_a for scalar case satisfies Klein–Gordon equation and for $x^\mu = (t,\mathbf{x})$ is defined[10] by Wightman function analytically continued[15] into the same future tube (V^+: $\zeta_a^0 > 0$, $\zeta_a^2 > 0$) as:

$$\begin{aligned} F_{p_a x_a}(x) &= e^{-i(p_a x_a)}\int \frac{d^3q}{(2\pi)^3 2E_q}\,\phi^\sigma(\mathbf{q},\mathbf{p}_a)e^{-i(q(x-x_a))}\\ &= (-i)e^{-i(p_a x_a)}N_\sigma D^-_{m_a}(x-x_a-i\zeta_a(p_a,\sigma_a))\,, \qquad g_2\equiv 0\,. \end{aligned} \tag{51}$$

This wave packet conforms with general requirements of quantum field theory[15] and due to (50) admits adequate description[10] of both the limits to the states localized in momentum and coordinate spaces at $\sigma_a \to 0$ and $\sigma_a\to\infty$, respectively:

$$\phi^\sigma(\mathbf{q},\mathbf{p}_a) \underset{\sigma\to 0}{\longmapsto} (2\pi)^3\, 2E_q\delta_3(\mathbf{q}-\mathbf{p}_a)\,, \qquad \phi^\sigma(\mathbf{q},\mathbf{p}_a) \underset{\sigma\to\infty}{\longmapsto} N_\infty\,. \tag{52}$$

Furthermore, because $2(qp_a) = 2m_a^2-(q-p_a)^2$, the non-relativistic limit of (49) and (51) for $g_1 = \sigma_a^{-2}$ exactly reproduces[10] the usual Gaussian profiles in coordinate and momentum representations independently of normalization arbitrariness (50).

To get the most simple example preserving oscillations we take an overlap function with one wave packet (49) and (51) in the one vertex representing the other vertex by infinitely heavy nucleus in the spirit of Kobzarev model.[16] For antineutrino this may be achieved, for example, in the Grimus–Stockinger model[4] with nucleons in creation vertex $\{C\}$ fixed in infinitely heavy nucleus and the bound state of initial electron in detection vertex $\{D\}$ replaced by above wave packet state for free electron. For the case of neutrino one can take the decay $\pi^+ \to \mu^+ + \nu_\mu$ with wave packet (49) for pion state and plane wave for muon in $\{C\}$-vertex taking $\{D\}$-vertex as in the Kobzarev model.[5,16] The overlap functions[5] $\mathcal{F}_{C,D}(-p)$ for the first case (compare with Eq. (11) from Ref. 4 for $\tilde{J}_\lambda = \text{const.}$), with $E_w = \sqrt{\mathbf{w}^2 + m_e^2}$, are reduced to:

$$\mathcal{F}_C(-p) = \text{const.}\,\delta(p^0 - Q_C^0)\,, \qquad Q_C^0 = \Delta M - U^0\,, \qquad Q_D = K - W\,, \tag{53}$$

$$\begin{gathered}\mathcal{F}_D(-p) = (2\pi)^4\delta(p^0 + E_w - K^0)\frac{\phi_e^\sigma(\mathbf{w},\mathbf{W})}{2E_w}\,,\\ \mathbf{w} = \mathbf{K} - \mathbf{p}\,, \qquad W^0 = E_W\,,\end{gathered} \tag{54}$$

where ΔM is neutron–proton mass difference, $U^\mu = (U^0, \mathbf{U})$ is four-momentum of created electron in vertex $\{C\}$; $W^\mu = (W^0, \mathbf{W})$ is the most probable four-momentum of incoming initial electron with $\zeta_e(m_e, \sigma_e) = g_1 W$ in Eq. (49), $w^\mu = (E_w, \mathbf{w})$, and $K^\mu = (K^0, \mathbf{K})$ is the total outgoing four-momentum carried away by final electron with final antineutrino in vertex $\{D\}$. Note that three-vectors $\mathbf{U}$ and $\mathbf{Q}_C$ become indefinite for this case. The product of these functions belongs to $S(\mathbf{R}^3_\mathbf{p})$ only for replaced argument of second delta-function with $E_w \mapsto E_W$, which appears naturally in sharply peak approximation for $|\mathbf{p} - \mathbf{Q}_D| = |\mathbf{W} - \mathbf{w}| \ll m_e$, leading to:

$$\mathcal{F}(-p) \approx \text{const.}\,\delta(p^0 - Q_C^0)\delta(p^0 - Q_D^0)\frac{N_{\sigma_e}}{2W^0}e^{-g_1(wW)}\,, \text{ where } g_1 = \frac{1}{\sigma_e^2}\,, \tag{55}$$

$$(wW) = E_\mathbf{w}W^0 - (\mathbf{w}\cdot\mathbf{W}) \approx m_e^2 + \frac{1}{2}((\mathbf{p} - \mathbf{Q}_D)^j(\delta^{jl} - \mathrm{V}^j\mathrm{V}^l)(\mathbf{p} - \mathbf{Q}_D)^l) \tag{56}$$

$$\begin{aligned}&= m_e^2 + \frac{1}{2}\{\mathbf{Q}_D^2 - (\mathbf{Q}_D\cdot\mathbf{V})^2 + \mathbf{p}^2\\ &\quad - 2(\mathbf{p}\cdot[\mathbf{Q}_D - \mathbf{V}(\mathbf{Q}_D\cdot\mathbf{V})]) - \mathbf{V}^2(\mathbf{pBp})\}\,,\end{aligned} \tag{57}$$

for

$$\begin{gathered}\mathbf{V} = \frac{\mathbf{W}}{W^0} = |\mathbf{V}|\boldsymbol{\omega}\,, \qquad \boldsymbol{\omega}^2 = 1\,,\\ B^{jl} = \omega^j\omega^l\,, \qquad B_0^{jl} = \omega^j\omega^l - \frac{\delta^{il}}{3}\,,\end{gathered} \tag{58}$$

with vector $\mathbf{V}$ as a most probable velocity of initial electron in detector. To define the coefficients $\Upsilon_s(k,\mathbf{n})$ (44) of asymptotic expansion (42) by making use of Eqs. (31)–(33), only the last two summands in the line (57) are now necessary, giving exactly the discussed here structure of the function $\Phi(\mathbf{q})$ (where due to the absence of vector $\mathbf{Q}_C$ its role partially plays vector $\mathbf{V}$). For non-relativistic initial electron $|\mathbf{V}| \ll 1$, and neglecting this velocity, for $\mathbf{Q}_D = |\mathbf{Q}_D|\boldsymbol{\rho}$ and real $\lambda = g_1 k|\mathbf{Q}_D| > 0$, now it is enough to take (see Appendix B for details and another cases):

$$\Phi(\mathbf{q}) = \Phi(-\mathbf{p}) = \Phi(-k\mathbf{n}) \underset{|\mathbf{V}|\ll 1}{\longmapsto} e^{\lambda\xi}\,, \qquad \xi = (\boldsymbol{\rho}\cdot\mathbf{n}) = \cos\Theta\,, \tag{59}$$

whence

$$C_1(k,\mathbf{n}) = 2\lambda\xi - \lambda^2(1-\xi^2)\,, \qquad \Upsilon_1(k,\mathbf{n}) = -4\lambda^2[1-(2+\lambda\xi)(1-\xi^2)]\,. \quad (60)$$

So $\Upsilon_1 < 0$, if $\lambda\xi < [(1-\xi^2)^{-1} - 2]$, that $\forall\,\lambda > 0$ includes interval $3\pi/4 \le \Theta \le \pi$,

$$\text{which for } \lambda \to \infty\,, \quad \text{dilates to } 0 \le \Theta \le 1/\sqrt{\lambda},\ \pi/2 + 1/\lambda \le \Theta \le \pi\,. \quad (61)$$

With the same function (59) $\forall\,\lambda < 0$ this condition implies interval $0 \le \Theta \le \pi/4$,

$$\text{dilating to } 0 \le \Theta \le \pi/2 - 1/|\lambda|\,,\ \pi - 1/\sqrt{|\lambda|} \le \Theta \le \pi\,, \ \text{ for } |\lambda| \to \infty\,, \quad (62)$$

that means the formal interchange in (61) of forward and backward hemispheres $\Theta \mapsto \pi - \Theta$. Similarly for the above mentioned case with neutrino, one can obtain for $\Phi(\mathbf{q}) = \Phi(k\mathbf{n})$ the same result (59) with $\lambda > 0$. Therefore, for both the cases $\Upsilon_1(k,\mathbf{n}) < 0$ near the forward direction (61) with the narrow wave packet $\sigma_{e,\pi} \to 0$ and/or with the high energy $p^0 = Q_D^0$ of anti-neutrino. Since due to Eq. (56), only this region in fact contributes to the averaging relative to $\boldsymbol{\rho}$, from (60) for the parameter ϱ_0 (48), with $k = \sqrt{(Q_D^0)^2 - m_j^2} \approx |\mathbf{Q}_D|$, and $Q_D^0 = Q_C^0$ (53), one finds:

$$\bar{\Upsilon}_1 \simeq -4\lambda^2\,, \qquad \varrho_0^2 \simeq \frac{4\lambda^2}{(2k)^2} = \frac{\mathbf{Q}_D^2}{\sigma_e^4}\,, \qquad \varrho_0 \approx \frac{k}{\sigma_e^2}\,, \quad (63)$$

$$\text{where} \qquad m_j \le \sigma_e \ll m_e\,, \qquad k \le Q_C^0 \le \Delta M - m_e\,. \quad (64)$$

For $k \simeq \Delta M - m_e \approx 0.783$ MeV and $\sigma_e \sim m_j \simeq 0.2$ eV it follows $\varrho_0 \simeq 3.86$ m. Note that for real β-decay of heavy nucleus the value of antineutrino energy $Q_C^0 \simeq k$ may be in many times greater. This qualitative analytical estimation illustrates and elucidates the numerical calculations[2,3] with more detailed models, that give the best value for this parameter $\varrho_0 \simeq 3.5$ m. The obtained analytical results (31)–(33), (42) and (44), together with relations from Appendix B can essentially simplify such calculations making them probably more transparent and model-independent.

5. Conclusions

As shown in Appendix A, following the same way as in Lemma 1 the operator expansion (20) for the Green function of Helmholtz equation may be generalized for arbitrary N-dimensional Euclidean space. The same concerns the assertion of Theorem 1 about asymptotic expansion for N-dimensional analog of integral (1).

The following comments are in order. Since the main principles of quantum field theory[15] require the wave packets belong to $S(\mathbf{R}^3_{\mathbf{q}})$, the same holds true for their convolution and thus for $\Phi(\mathbf{q})$, as was supposed in Theorem 1. This property ensures the possibility of further asymptotic expansion (31) and (37) of integral (1) and defines the order of leading corrections to Grimus–Stockinger asymptotic,[4] which is crucial for explanation[2,3] (48) of (anti-) neutrino deficit.

For electron in stationary bound state[4] with energy $W^0 = E_b$ instead of the wave packet state, the overlap function (54) with the also indefinite now vectors

$\mathbf{W}$, $\mathbf{K}$ and $\mathbf{Q}_D$ becomes: $\mathcal{F}_D(-p) = \delta(p^0 + E_b - K^0)\tilde{\psi}_b(\mathbf{w}) \in C^\infty(\mathbf{R}^3_\mathbf{p})$, where $|\tilde{\psi}_b(\mathbf{K}-\mathbf{p})| \leq O(|\mathbf{p}|^{-4})$ for $|\mathbf{p}| \to \infty$. It may be argued following Lemma 1 of Ref. 2 and Theorem of Ref. 4, that this fourth power fall-off should be enough for the well definiteness of the asymptotic expansion (31) up to the term with $s = 2$, and therefore the expansion (42) and (44) up to the term with $n = 1$, preserves Eqs. (47) and (48). On the other hand, the infinite coherence length of neutrino oscillations for the stationary case[4] is also the price for the absence of external wave packets like (49) and (51); only their presence in both the creation $\{C\}$ and detection $\{D\}$ vertices provides a finite value of coherence length.[5,6] So we conclude, that again as in Ref. 4 "Therefore the physics under consideration seems to comply naturally with the mathematical requirements".

Since the wave packet (49) is in momentum representation $\phi^\sigma(\mathbf{q}, \mathbf{p}_a) \in S(\mathbf{R}^3_\mathbf{q})$, the same takes place[15] for its coordinate representation (51). So, for fixed t, $F_{p_a x_a}(t, \mathbf{x}) \in S(\mathbf{R}^3_\mathbf{x})$ obviously with infinite support and the same holds true for their convolution $\phi(\mathbf{x})$ in Eqs. (36) and (12). Therefore, the expansion (37) acquires also the additional asymptotic nature, with nonzero corrections (38), governed by detailed form of asymptotic behavior of the function $\phi(\mathbf{x})$ itself. These corrections inevitably introduce an additional dimensional parameter and may also affect the behavior of $\mathcal{J}(\mathbf{R})$ at microscopically big but macroscopically small distances R, discussed here and in Refs. 2, 3 and 9.

Appendix A

For N-dimensional space, defining $\nu = (N-2)/2$, $a = (N-3)/2 = \nu - \frac{1}{2}$, $l = 0, 1, 2, \ldots$, $j = l + a$, with $l(l+2\nu) + \nu^2 - \frac{1}{4} = l(l+2\nu) + a(a+1) = j(j+1)$, for $\mathbf{R} = R\mathbf{n}$, $\mathbf{x} = r\mathbf{v}$, $\mathbf{k} = k\mathbf{n}$, one has[12,13] Gegenbauer polynomials $C_l^{(\nu)}((\mathbf{n}\cdot\mathbf{v}))$ as eigenfunction of N-dimensional square angular momenta operator $\mathcal{L}^{(N)}_\mathbf{n}$, where for $\boldsymbol{\nabla}_\mathbf{R}$ in (4), instead of (5) and (6): $[(\boldsymbol{\partial}_\mathbf{n})_\beta, n_\alpha] = \delta_{\alpha\beta} - n_\alpha n_\beta$, $(\mathbf{n}\cdot\boldsymbol{\partial}_\mathbf{n}) = 0$, $(\boldsymbol{\partial}_\mathbf{n}\cdot\mathbf{n}) = N-1$, $x_\alpha\partial_\beta - x_\beta\partial_\alpha = n_\alpha(\boldsymbol{\partial}_\mathbf{n})_\beta - n_\beta(\boldsymbol{\partial}_\mathbf{n})_\alpha = i\mathrm{L}_{\alpha\beta}$ and $-\boldsymbol{\partial}^2_\mathbf{n} = \mathbf{L}^2/2 = \mathcal{L}^{(N)}_\mathbf{n}$, so that:[13]

$$\boldsymbol{\nabla}^2_\mathbf{R} = \partial^2_R + \frac{N-1}{R}\partial_R - \frac{\mathcal{L}^{(N)}_\mathbf{n}}{R^2} = \frac{1}{R^{a+1}}\partial^2_R R^{a+1} - \frac{a(a+1)}{R^2} - \frac{\mathcal{L}^{(N)}_\mathbf{n}}{R^2}\,, \quad \text{(A.1)}$$

$$\mathcal{L}^{(N)}_\mathbf{n} C_l^{(\nu)}((\mathbf{n}\cdot\mathbf{v})) = l(l+2\nu)C_l^{(\nu)}((\mathbf{n}\cdot\mathbf{v}))\,. \quad \text{(A.2)}$$

In definitions (27) and (28), where,[8,11–13] for $l \mapsto j$, with $|\arg z - \beta_{1,2}| < \pi/2$:

$$K_\lambda(z) = \frac{1}{2}\int_{0e^{i\beta_2}}^{\infty e^{-i\beta_1}} \frac{dt}{t} t^{\pm\lambda} \exp\left\{-\frac{z}{2}\left(t + \frac{1}{t}\right)\right\}, \quad \text{(A.3)}$$

and

$$\int_0^\infty dr\, \psi_{j\,0}(kr)\psi_{j\,0}(qr) = \frac{\pi}{2}\delta(q-k)\,, \quad \text{(A.4)}$$

according to formulas (8.534) and (8.532) of Ref. 12 ($H_\nu^{(1),(2)}(z)$ are the Hankel functions,[8,12]) the plane wave and the Green function now read:[13,17]

$$e^{\mp i(\mathbf{k}\cdot\mathbf{x})} = \sqrt{\frac{2}{\pi}}\frac{2^{\nu-1}\Gamma(\nu)}{(\mp ikr)^a kr}\sum_{j=a}^{\infty}(2j+1)i^{\mp j}\psi_{j\,0}(kr)C_l^{(\nu)}((\mathbf{n}\cdot\mathbf{v}))\,, \tag{A.5}$$

$$\left\langle\mathbf{R}\left|\frac{1}{-\boldsymbol{\nabla}^2-k^2\mp i0}\right|\mathbf{x}\right\rangle = \int\frac{d^Nq}{(2\pi)^N}\frac{e^{i(\mathbf{q}\cdot(\mathbf{R}-\mathbf{x}))}}{(\mathbf{q}^2-k^2\mp i0)} \tag{A.6}$$

$$= \pm\frac{i}{4}\left(\frac{k}{2\pi}\right)^{\nu}\frac{H_\nu^{\substack{(1)\\(2)}}(k|\mathbf{R}-\mathbf{x}|)}{|\mathbf{R}-\mathbf{x}|^{\nu}} = \sqrt{\frac{\pi}{2}}\frac{(\mp ik)^a}{(2\pi)^{\nu+1}}\frac{\chi_a(\mp ik|\mathbf{R}-\mathbf{x}|)}{|\mathbf{R}-\mathbf{x}|^{a+1}} \tag{A.7}$$

$$= \frac{\Gamma(\nu)}{4\pi^{\nu+1}}\frac{1}{(Rr)^{a+1}}\sum_{j=a}^{\infty}(2j+1)\frac{i^{\mp j}}{k}\chi_j(\mp ikR)\psi_{j0}(kr)C_l^{(\nu)}((\mathbf{n}\cdot\mathbf{v}))\,, \tag{A.8}$$

for $R > r$ and $l = j-a$, where a and $j \geq a$ are both integer for odd N or half integer for even N. So, for $R > r$, with again really not appearing operator $l \mapsto \Lambda_{\mathbf{n}}^{(N)}$, one obtains similarly the following generalization of (19) and (20) of Lemma 1:

$$\sqrt{\frac{\pi}{2}}\frac{(\mp ik)^a}{(2\pi)^{\nu+1}}\frac{\chi_a(\mp ik|\mathbf{R}-\mathbf{x}|)}{|\mathbf{R}-\mathbf{x}|^{a+1}} = \sqrt{\frac{\pi}{2}}\frac{(\mp ik)^a}{(2\pi)^{\nu+1}}\frac{\chi_{\Lambda_{\mathbf{n}}^{(N)}+a}(\mp ikR)}{R^{a+1}}e^{\mp ik(\mathbf{n}\cdot\mathbf{x})} \tag{A.9}$$

$$\sim \sqrt{\frac{\pi}{2}}\frac{(\mp ik)^a}{(2\pi)^{\nu+1}}\frac{e^{\pm ikR}}{R^{a+1}}\left\{1+\sum_{s=1}^{\infty}\frac{\prod_{\mu=1}^{s}[\mathcal{L}_{\mathbf{n}}^{(N)}+a(a+1)-\mu(\mu-1)]}{s!(\mp 2ikR)^s}\right\}e^{\mp ik(\mathbf{n}\cdot\mathbf{x})}\,. \tag{A.10}$$

Here, (A.9) is again an operator rewritten of (A.8) via (A.5) and positively defined self-adjoint operator with positive eigenvalue, from which $\chi_j(z)$ (27) depends again like the whole function (A.3), with:

$$\mathcal{L}_{\mathbf{n}}^{(N)} + a(a+1) + \frac{1}{4} = \left(\Lambda_{\mathbf{n}}^{(N)} + a + \frac{1}{2}\right)^2 \longmapsto \left(j+\frac{1}{2}\right)^2 \tag{A.11}$$

and again the asymptotic version[8,12] of expansion (27) is used. Due to (A.1), instead of operator $\mathcal{L}_{\mathbf{n}}^{(N)} + a(a+1)$ in (A.10), we may again use the operator from the square brackets of the radial Schrödinger equation:

$$\left[r^2\left(\frac{1}{r^{a+1}}\partial_r^2 r^{a+1} + k^2\right)\right]\frac{\psi_{j\,0}(kr)}{r^{a+1}} = j(j+1)\frac{\psi_{j0}(kr)}{r^{a+1}}\,. \tag{A.12}$$

Appendix B

The differential operators $\boldsymbol{\partial}_{\mathbf{n}}$ and $\mathcal{L}_{\mathbf{n}} = -\boldsymbol{\partial}_{\mathbf{n}}^2$ defined in Eqs. (4)–(7) have the following properties for any differentiable scalar functions $f(\xi)$, $\mathcal{W}(\xi,\varsigma)$, $\mathcal{H}(\zeta)$ with ξ from Eq. (59), $\varsigma = (\boldsymbol{\varrho}\cdot\mathbf{n})$, $\zeta_m = (\mathbf{n}\mathbf{B}^m\mathbf{n})$, $\zeta \equiv \zeta_1$, $\mathrm{Tr}\{\mathbf{B}^m\} = 3\bar{\zeta}_m$, $\zeta_{m0} = \zeta_m - \bar{\zeta}_m$:

$$\boldsymbol{\partial}_{\mathbf{n}} f(\xi) = \boldsymbol{\partial}_{\mathbf{n}} f((\boldsymbol{\rho}\cdot\mathbf{n})) = \boldsymbol{\rho}_\perp \partial_\xi f(\xi) + f(\xi)\boldsymbol{\partial}_{\mathbf{n}}\,, \qquad \boldsymbol{\rho}_\perp = \boldsymbol{\rho} - \mathbf{n}(\boldsymbol{\rho}\cdot\mathbf{n})\,, \quad \text{(B.1)}$$

$$(\boldsymbol{\partial}_{\mathbf{n}}\xi) = \boldsymbol{\rho}_\perp\,, \qquad (\boldsymbol{\partial}_{\mathbf{n}}\varsigma) = \boldsymbol{\varrho}_\perp\,, \qquad ((\boldsymbol{\partial}_{\mathbf{n}})^j\zeta_m) = 2[(\mathbf{B}^m\mathbf{n})^j - n^j\zeta_m]\,, \quad \text{(B.2)}$$

$$\mathcal{L}_{\mathbf{n}}\xi = 2\xi\,, \qquad \mathcal{L}_{\mathbf{n}}\varsigma = 2\varsigma\,, \qquad \mathcal{L}_{\mathbf{n}}\zeta_m = 2[3\zeta_m - \mathrm{Tr}\{\mathbf{B}^m\}] \equiv 6\zeta_{m0} = \mathcal{L}_{\mathbf{n}}\zeta_{m0}\,, \quad \text{(B.3)}$$

$$f(\xi)C_1(\xi) = \mathcal{L}_{\mathbf{n}} f(\xi) = \partial_\xi(\xi^2-1)\partial_\xi f(\xi) = [(\xi^2-1)\partial^2_{\xi^2} + 2\xi\partial_\xi]f(\xi)\,, \quad \text{(B.4)}$$

$$f(\xi)C_2(\xi) = \left[(6\xi^2-2)\partial^2_{\xi^2} + 4\xi(\xi^2-1)\partial^3_{\xi^3} + \frac{1}{2}(\xi^2-1)^2\partial^4_{\xi^4}\right]f(\xi)\,, \quad \text{(B.5)}$$

$$\mathcal{L}_{\mathbf{n}}\mathcal{W}(\xi,\varsigma) = [\partial_\xi(\xi^2-1)\partial_\xi + \partial_\varsigma(\varsigma^2-1)\partial_\varsigma + 2(\xi\varsigma - (\boldsymbol{\varrho}\cdot\boldsymbol{\rho}))\partial^2_{\xi\varsigma}]\mathcal{W}(\xi,\varsigma)\,, \quad \text{(B.6)}$$

$$\mathcal{L}_{\mathbf{n}} f(\xi)g(\varsigma) = g(\varsigma)\mathcal{L}_{\mathbf{n}} f(\xi) + f(\xi)\mathcal{L}_{\mathbf{n}} g(\varsigma) - 2(\boldsymbol{\rho}_\perp\cdot\boldsymbol{\varrho}_\perp)\partial_\xi f(\xi)\partial_\varsigma g(\varsigma)\,, \quad \text{(B.7)}$$

$$\mathcal{H}(\zeta)C_1(\zeta) = \mathcal{L}_{\mathbf{n}}\mathcal{H}(\zeta) = \{6\zeta_0\partial_\zeta + 4(\zeta^2-\zeta_2)\partial^2_{\zeta^2}\}\mathcal{H}(\zeta)\,, \qquad \zeta_0 \equiv \zeta_{10}\,, \quad \text{(B.8)}$$

$$\begin{aligned}\mathcal{H}(\zeta)C_2(\zeta) &= \frac{1}{2}[\mathcal{L}^2_{\mathbf{n}} - 2\mathcal{L}_{\mathbf{n}}]\mathcal{H}(\zeta) && \text{(B.9)}\\ &= \{12\zeta_0\partial_\zeta + 6[(4\zeta+3\zeta_0)\zeta_0 + 6(\zeta^2-\zeta_2) - 2\zeta_{20}]\partial^2_{\zeta^2}\\ &\quad + 8[(4\zeta+3\zeta_0)(\zeta^2-\zeta_2) + 2(\zeta_3-\zeta\zeta_2)]\partial^3_{\zeta^3}\\ &\quad + 8(\zeta^2-\zeta_2)^2\partial^4_{\zeta^4}\}\mathcal{H}(\zeta)\,. && \text{(B.10)}\end{aligned}$$

When $f(\xi)\mapsto e^{\lambda\xi}$, Eqs. (B.4) and (B.5) with $\partial_\xi\mapsto\lambda$ immediately give Eq. (60). Similarly for $\mathcal{H}(\zeta)\mapsto e^{\lambda\zeta}$ one has $\partial_\zeta\mapsto\lambda$ in Eqs. (B.8)–(B.10), leading to:

$$\Upsilon_1 = -8\lambda[a\lambda^2 + b\lambda + c]\,, \qquad \text{with} \;\; a = 4[2\zeta^3 + \zeta_3 - 3\zeta\zeta_2]\,, \quad \text{(B.11)}$$

$$\begin{aligned} b &= 3[2\zeta_0\zeta + 3(\zeta^2-\zeta_2) - \zeta_{20}]\,,\\ c &= 3\zeta_0\,, \qquad \zeta_m = (\mathbf{n}\mathbf{B}^\sigma\mathbf{B}^{m-\sigma}\mathbf{n}) > 0\,, \end{aligned} \quad \text{(B.12)}$$

$$\begin{aligned} &(\zeta_m)^2 \le \zeta_{2\sigma}\zeta_{2m-2\sigma}\,,\\ &\zeta_m \ge \zeta\zeta_{m-1} \ge \zeta^2\zeta_{m-2} \ge \cdots \ge \zeta^k\zeta_{m-k}\cdots \ge \zeta^m\,, \end{aligned} \quad \text{(B.13)}$$

due to positive definiteness of $\mathbf{B}$ and Cauchy–Schwarz inequality. Formally the case (B.11) and (B.12) appears in Eq. (57) for "collinear" events with $\mathbf{K}\parallel\mathbf{V}$ only, for $\lambda = g_1k^2\mathbf{V}^2/2 > 0$ and degenerate tensor $\mathbf{B}$ (58) evidently realizing all inequalities (B.13). The structure of $\mathbf{B}$ and λ becomes much more complicated for the complete models.[2,6] Nevertheless, the most general case containing all variables simultaneously, like $\Phi(\mp k\mathbf{n})\mapsto e^{\alpha\xi}e^{\beta\varsigma}e^{\lambda\zeta}$, may be considered by the same way using the above relations together.

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