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DISAPPEARANCE OF SCHWINGER'S STRING AT THE CHARGE–MONOPOLE "MOLECULE"

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An equivalence of total angular momentum operator of charge–monopole system to the momentum operator of a symmetrical quantum top is observed. This explicitly shows the string independence of Dirac's quantization condition leading to disappearance of Schwinger's string and reveals some properties of diatomic molecule for this system.

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1. Introduction

It is well known[1,2] that the one-particle wave function of a charge scattered on monopole's field decomposes as a sum of rotation eigenfunctions of symmetrical quantum top. Here some physical reasons of this will be elucidated.

Let us recall the main results about the problem under consideration (see Refs. 3, 4 and references therein). It is well known that magnetic field's "hedgehog" $\mathbf{B}_m(\mathbf{x})$ of an infinitely heavy magnetic monopole placed at the origin O and corresponded to residual (Abelian) U(1) gauge group has a source at the origin O viewed from classical point either as the origin of a semi-infinite infinitely thin solenoid (bar magnet string) $\mathbf{A}^{\pm}_{\mathbf{u}}(\mathbf{x})$ along the direction of vector $\mathbf{u}$ (Dirac),[5] or as the origin of two such symmetric strings $\mathbf{A}_{\mathbf{u}}(\mathbf{x})$ (Schwinger)[6]:

$$\mathbf{A}^{\pm}_{\mathbf{u}}(\mathbf{x}) = \frac{g}{r}\frac{(\mathbf{u}\times\mathbf{x})}{((\mathbf{u}\cdot\mathbf{x})\pm r)}\,, \qquad \mathbf{A}_{\mathbf{u}}(\mathbf{x}) = \frac{1}{2}(\mathbf{A}^{+}_{\mathbf{u}}(\mathbf{x}) + \mathbf{A}^{-}_{\mathbf{u}}(\mathbf{x}))\,, \tag{1}$$

$$\mathbf{B}_m(\mathbf{x}) = (\boldsymbol{\nabla}_{\mathrm{x}}\times\mathbf{A}_{\mathbf{u}}(\mathbf{x})) - \mathbf{h}_{\mathbf{u}}(\mathbf{x}) = g\frac{\mathbf{x}}{r^3}\,, \qquad r = |\mathbf{x}|\,, \tag{2}$$

$$\mathbf{h}_{\mathbf{u}}(\mathbf{x}) = -2\pi g\mathbf{u}\frac{(\mathbf{u}\cdot\mathbf{x})}{r}\delta_{2\mathbf{u}}(\mathbf{x}_{\perp})\,, \qquad \mathbf{x}_{\perp} = \mathbf{x} - \mathbf{u}(\mathbf{u}\cdot\mathbf{x})\,, \tag{3}$$

for which:
$$(\boldsymbol{\nabla}_{\rm x}\cdot\mathbf{A}^{\pm}_{\mathbf{u}}(\mathbf{x}))=(\boldsymbol{\nabla}_{\rm x}\cdot\mathbf{A}_{\mathbf{u}}(\mathbf{x}))=0\,,\tag{4}$$
$$(\boldsymbol{\nabla}_{\rm x}\cdot\mathbf{B}_{m}(\mathbf{x}))=-(\boldsymbol{\nabla}_{\rm x}\cdot\mathbf{h}(\mathbf{x}))=4\pi g\delta_3(\mathbf{x})\,,\tag{5}$$

where $\mathbf{h}_{\mathbf{u}}(\mathbf{x})$ is the magnetic field inside the string, defined by two-dimensional $\delta_{2\mathbf{u}}(\mathbf{x}_{\perp})$-function. The same magnetic field (2) results[3] also from $\mathbf{A}^{\pm}_{\mathbf{u}}(\mathbf{x})$ with corresponding $\mathbf{h}^{\pm}_{\mathbf{u}}(\mathbf{x})$.

In Cartesian basis $\mathbf{e}_i$ placed at the origin O for a charge position vector $\mathbf{x}=r\mathbf{n}=\rho\boldsymbol{\eta}_{(\rho)}+z\mathbf{e}_3$ one has the rotating vectors of spherical and polar bases $\boldsymbol{\eta}_{(j)}(\beta,\alpha)$ as functions of the corresponding angles α, β:

$$\mathbf{n}=\boldsymbol{\eta}_{(r)}=\mathbf{e}_1\sin\beta\cos\alpha+\mathbf{e}_2\sin\beta\sin\alpha+\mathbf{e}_3\cos\beta\,,\tag{6}$$
$$\boldsymbol{\eta}_{(\beta)}=\mathbf{e}_1\cos\beta\cos\alpha+\mathbf{e}_2\cos\beta\sin\alpha-\mathbf{e}_3\sin\beta\,,\tag{7}$$
$$\boldsymbol{\eta}_{(\alpha)}=-\mathbf{e}_1\sin\alpha+\mathbf{e}_2\cos\alpha\,,\tag{8}$$
$$\boldsymbol{\eta}_{(\rho)}=\mathbf{e}_1\cos\alpha+\mathbf{e}_2\sin\alpha=\boldsymbol{\eta}_{(\beta)}\cos\beta+\mathbf{n}\sin\beta\,.\tag{9}$$

Thus, the gauge $\mathbf{u}=\mathbf{e}_3=\mathbf{e}_z$, with $(\mathbf{u}\cdot\mathbf{n})=\cos\beta$, $(\mathbf{u}\times\mathbf{n})=\boldsymbol{\eta}_{(\alpha)}\sin\beta$, recasts the different fields of Eq. (1) to the following:

$$\mathbf{A}^{+}_{\mathbf{u}}(\mathbf{x})=\boldsymbol{\eta}_{(\alpha)}\frac{g}{r}\tan\frac{\beta}{2}\,,\qquad\mathbf{A}^{-}_{\mathbf{u}}(\mathbf{x})=-\boldsymbol{\eta}_{(\alpha)}\frac{g}{r}\cot\frac{\beta}{2}\,,\tag{10}$$
$$\mathbf{A}_{\mathbf{u}}(\mathbf{x})=-\boldsymbol{\eta}_{(\alpha)}\frac{g}{r}\cot\beta\,.\tag{11}$$

Here the first and second ones are for the semi-infinite (Dirac) strings along $-\mathbf{e}_z$ and $\mathbf{e}_z$ respectively, whereas the third one is for the infinite (Schwinger) string, composed symmetrically by the two previous ones.[3]

Various expressions (1) for $\mathbf{A}_{\mathbf{u}}(\mathbf{x})$ are differed by the gauge transformation containing a multivalued gauge function[4] $\Lambda(\mathbf{x})$. For example for transfer (rotation) from the semi-infinite string along $-\mathbf{e}_z$ of Eq. (10) to the one along the $\mathbf{e}_z$, this transformation[3,4] is a gauge one only "almost everywhere", out of the semi-infinite half plane, $y=0$, $x>0$, bounded by infinite z-axis. The potentials (10) lead to[3]:

$$\frac{e}{c\hbar}(\mathbf{A}^{+}_{\mathbf{u}}(\mathbf{x})-\mathbf{A}^{-}_{\mathbf{u}}(\mathbf{x}))=\boldsymbol{\nabla}_{\rm x}\Lambda(\mathbf{x})=\frac{2Q}{\hbar}\frac{\boldsymbol{\eta}_{(\alpha)}}{r\sin\beta}\,,\tag{12}$$

for:
$$\boldsymbol{\nabla}_{\rm x}=\mathbf{n}\frac{\partial}{\partial r}+\frac{\boldsymbol{\eta}_{(\beta)}}{r}\frac{\partial}{\partial\beta}+\frac{\boldsymbol{\eta}_{(\alpha)}}{r\sin\beta}\frac{\partial}{\partial\alpha}\,,\qquad\Lambda(\mathbf{x})=\frac{2Q}{\hbar}\alpha\,,\tag{13}$$

with:
$$Q=\frac{eg}{c}\,,\quad\text{and give: }\boldsymbol{\pi}_{+}=e^{i\Lambda(\mathbf{x})}\boldsymbol{\pi}_{-}e^{-i\Lambda(\mathbf{x})}\,,\tag{14}$$

for:
$$\boldsymbol{\pi}_{\pm}=\mathbf{p}-\frac{e}{c}\mathbf{A}^{\pm}_{\mathbf{u}}(\mathbf{x})\,,\qquad\mathbf{p}=-i\hbar\boldsymbol{\nabla}_{\rm x}\,.\tag{15}$$

Single-valuedness of $e^{i\Lambda(\mathbf{x})}$ for $\alpha \to \alpha + 2\pi$ imposes Dirac's quantization conditions[5]:

$$e^{i\Lambda(\mathbf{x})} = e^{iN\alpha}\,, \qquad -2Q = \hbar N\,, \qquad N = 0, \pm 1, \pm 2, \dots\,. \tag{16}$$

Schwinger's symmetrical string of Eq. (11) seems believable to lead to a more restrictive condition[6] with $N \mapsto 2N$ only. However, this string possess another interpretation without such a restriction, as it will be shown below.

On the other hand, a common classical electromagnetic field (EMF) composing by the magnetic field $\mathbf{B}_m(\mathbf{y})$ from the monopole at the origin O and by the electric field $\mathbf{E}_e(\mathbf{y})$ from the scattered point charge at the position $\mathbf{x}$, brings into their system an additional irremovable angular momentum $\mathbf{M} = -Q\mathbf{n}$, associated with Poynting momentum density vector.[1,3] In spite of the impossibility to assign this angular momentum to any one of these particles, it has inspired Goldhaber[1] and others[2,3] to interpret the value $-Q$ quantized by Eq. (16) as a conserving projection onto the vector $\mathbf{n}$ of some additional quantum spin $\mathbf{S}$ satisfying the relations:

$$(\mathbf{M}\cdot\mathbf{n}) = -Q \leftrightarrow (\mathbf{S}\cdot\mathbf{n})\,, \qquad (\mathbf{S}\cdot\mathbf{n}) \mapsto \hbar\mu\,, \qquad 2\mu = N\,, \tag{17}$$

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k\,, \qquad [\mathbf{S}, x_j] = [\mathbf{S}, p_j] = 0\,, \tag{18}$$

$$[(\mathbf{S}\cdot\mathbf{n}), \boldsymbol{\pi}_s] = 0\,, \quad \text{where for } \boldsymbol{\pi}_s\text{: } \frac{e}{c}\mathbf{A}_s(\mathbf{x}) = -\frac{(\mathbf{S}\times\mathbf{x})}{r^2}\,, \tag{19}$$

with:

$$\frac{e}{c}\mathbf{B}_m(\mathbf{x}) = \frac{1}{i\hbar}(\boldsymbol{\pi}\times\boldsymbol{\pi}) = \frac{e}{c}(\boldsymbol{\nabla}_{\mathrm{x}} \times \mathbf{A}_s(\mathbf{x})) \tag{20}$$

$$-\frac{i}{\hbar}\left(\frac{e}{c}\mathbf{A}_s(\mathbf{x}) \times \frac{e}{c}\mathbf{A}_s(\mathbf{x})\right) = -(\mathbf{S}\cdot\mathbf{n})\frac{\mathbf{x}}{r^3}\,, \tag{21}$$

— instead of the string expression (2). Thus, he has avoided the use of any strings of Eq. (1). The spin quantization condition (17) is equivalent to (16) making these strings invisible. However, instead of the string potentials of Eq. (1), he has arrived at the non-Abelian spin-potential[3,4] $\mathbf{A}_s(\mathbf{x})$ given by Eq. (19), which also obeys Eq. (4) but is connected with the string ones $\mathbf{A}^{\pm}_{\mathbf{u}}(\mathbf{x})$ by a spin rotation that is meaningful only on the eigenstates of the third spin-component[2,3] S_3. A single spin is enough[1] to obtain Dirac's strings $\mathbf{A}^{\pm}_{\mathbf{u}}(\mathbf{x})$ only. To reproduce Schwinger's string $\mathbf{A}_{\mathbf{u}}(\mathbf{x})$ of Eq. (11), it is necessary to take the operator $\mathbf{S}$ as a sum of two mutually commutative spin operators,[3] $\mathbf{S} = \mathbf{S}_a + \mathbf{S}_b$. The final result is reached by using the unitary transformation:[3] $\mathrm{U} = \mathcal{U}_a^{-1}(\alpha, \beta, -\alpha)\mathcal{U}_b^{-1}(\alpha, \beta - \pi, -\alpha)$ (cf. Eq. (53) below), rotated the projections of these two spins on the vector $\mathbf{n}$ into their projections on the vectors $\pm\mathbf{e}_z$ respectively: $(\mathbf{S}_{a,b}\cdot\mathbf{n}) \mapsto \pm(\mathbf{S}_{a,b}\cdot\mathbf{e}_3) = \pm(\mathbf{S}_{a,b})_3$, and furnished by eigenvalue condition $(\mathbf{S}_a - \mathbf{S}_b)'_3 = -Q$. A crown of this cumbersome construction[2,3] is an impression that for $\boldsymbol{\mathcal{L}}_s \equiv (\mathbf{x}\times\boldsymbol{\pi}_s)$ and with the first substitution of Eq. (17) the total angular momentum operator $\boldsymbol{J}_s$ takes a simple — "one-particle" form[1] with the usual orbital momentum $\mathbf{L} = (\mathbf{x} \times \mathbf{p})$ and the spin $\mathbf{S}$:

$$\boldsymbol{J}_s = \boldsymbol{\mathcal{L}}_s - Q\mathbf{n} \leftrightarrow \boldsymbol{\mathcal{L}}_s + (\mathbf{S} \cdot \mathbf{n})\mathbf{n} = \mathbf{L} + \mathbf{S} \equiv \boldsymbol{\mathcal{J}} \,, \tag{22}$$

and that the above rotation converts it to the standard one[3]:

$$\mathrm{U}\boldsymbol{\pi}_s\mathrm{U}^{-1} = \boldsymbol{\pi} = \mathbf{p} + \frac{\boldsymbol{\eta}_{(\alpha)}}{r} \sin\beta\left(\frac{S'_{a3}}{1+\cos\beta} + \frac{S'_{b3}}{1-\cos\beta}\right), \tag{23}$$

$$\mathrm{U}\boldsymbol{\mathcal{J}}\mathrm{U}^{-1} = \boldsymbol{J} = (\mathbf{x} \times \boldsymbol{\pi}) + \mathbf{n}(\mathbf{S}_a - \mathbf{S}_b)'_3 \,. \tag{24}$$

Here $S'_{a3} = 0$ for the one of Dirac's string whereas for Schwinger's one $(\mathbf{S}_a + \mathbf{S}_b)'_3 = 0$. Since $(\mathbf{L} \cdot \mathbf{n}) = (\boldsymbol{\mathcal{L}} \cdot \mathbf{n}) = 0$, one has $(\mathbf{S} \cdot \mathbf{n}) = (\boldsymbol{\mathcal{J}} \cdot \mathbf{n})$, what also helps to convert the Hamiltonian operator $2mH_s = \boldsymbol{\pi}_s^2$ into the usual form[1]:

$$\mathrm{U}\boldsymbol{\pi}_s^2\mathrm{U}^{-1} = \boldsymbol{\pi}^2 = 2mH = \tilde{p}_r^2 + \frac{\boldsymbol{\mathcal{L}}^2}{r^2} \,. \tag{25}$$

Note that one of the summands in Eq. (1) always disappears as $\mathbf{n} \to \pm\mathbf{u}$ for the charge leaving "visible" only singular contribution of the other one, which is equal to 1/2 of Dirac's string as a "half" of Schwinger's string. This qualitatively makes very smooth a physical difference between two types of these strings, because by using a gauge-like transformation (12) an arbitrary position of (Schwinger's) string can always be directed almost along the vector $\mathbf{n}$ of incident charge without violating "Dirac's veto".[3]

The aim of the present letter is to show that the above one-particle interpretation of the total angular momentum operator can be replaced naturally by its interpretation for some extended object leading to disappearance of Schwinger's string.

2. The Algebra of Operators in the Presence of Monopole

Let us consider the algebra of operators for the motion of a charge in the monopole fields (1) with the Hamiltonian (25) and local commutation relations[1–4]:

$$[x_i, x_j] = 0 \,, \qquad [x_i, \pi_j] = i\hbar\delta_{ij} \,, \qquad [\pi_i, \pi_j] = i\frac{e\hbar}{c}\epsilon_{ijk}(\mathbf{B}_m)_k \,, \tag{26}$$

$$\frac{1}{2}\epsilon_{ijk}[\pi_i, [\pi_j, \pi_k]] = \frac{e\hbar^2}{c}(\boldsymbol{\nabla}_{\mathrm{x}} \cdot \mathbf{B}_m(\mathbf{x})) = 4\pi\hbar^2 Q\delta_3(\mathbf{x}) \,. \tag{27}$$

In terms of the operator $\boldsymbol{\mathcal{L}} = (\mathbf{x} \times \boldsymbol{\pi})$, the equation of motion takes its classical form[1]:

$$2mi\hbar\dot{\boldsymbol{\pi}} = [\boldsymbol{\pi}, \boldsymbol{\pi}^2] = i\frac{e\hbar}{c}((\boldsymbol{\pi} \times \mathbf{B}_m) - (\mathbf{B}_m \times \boldsymbol{\pi})) = \frac{2}{i}\hbar\frac{Q}{r^3}\boldsymbol{\mathcal{L}} \,, \tag{28}$$

$$2mi\hbar\dot{\boldsymbol{\mathcal{L}}} = [\boldsymbol{\mathcal{L}}, \boldsymbol{\pi}^2] = Q2mi\hbar\dot{\mathbf{n}} = Q[\mathbf{n}, \boldsymbol{\pi}^2] \,. \tag{29}$$

The substitution $-Q \leftrightarrow (\boldsymbol{J}\cdot\mathbf{n})$ instead of the first one of Eq. (17) for the angular momentum operator $\boldsymbol{J} = \boldsymbol{\mathcal{L}} - Q\mathbf{n} \leftrightarrow \boldsymbol{\mathcal{L}} + (\boldsymbol{J}\cdot\mathbf{n})\mathbf{n}$ instead of the $\boldsymbol{\mathcal{J}}$ in Eq. (22) gives[1–4,7]:

$$2mi\hbar\dot{\boldsymbol{J}} = [\boldsymbol{J}, \boldsymbol{\pi}^2] = 0\,, \qquad [(\boldsymbol{J}\cdot\mathbf{n}), \boldsymbol{\pi}] = 0\,, \qquad [(\boldsymbol{J}\cdot\mathbf{n}), \mathbf{n}] = 0\,, \tag{30}$$

$$[J_i, x_j] = i\hbar\epsilon_{ijk}x_k\,, \qquad [J_i, \pi_j] = i\hbar\epsilon_{ijk}\pi_k\,, \qquad [J_i, J_j] = i\hbar\epsilon_{ijk}J_k\,. \tag{31}$$

Hence $\boldsymbol{J}$ is a conserving total angular momentum operator for the extended system: "charge + monopole + common EMF", though the last two formulas in Eq. (31) are valid, strictly speaking, only outside of the string $\mathbf{h}_{\mathbf{u}}(\mathbf{x})$ producing an additional contribution[3] from Eq. (3). The first relation of Eq. (26) together with the first and the last relations of Eq. (31) form the algebra of Euclidean group E3 having $(\boldsymbol{J}\cdot\mathbf{n})$ as Casimir operator,[7] what leads to quantization condition (16) without any reference to explicit form of the potential $\mathbf{A}^{\pm}_{\mathbf{u}}(\mathbf{x})$. At last Jackiw[8] has showed recently that Jacobi commutator (27) provides the condition (16) under direct construction of extended object such as tetrahedron.

Hurst[9] was probably the first who used the differential form instead of the spin one (17) for the projection of the total angular momentum operator $(\boldsymbol{J}\cdot\mathbf{n})$ in the case of Dirac's potentials (10). Remaining, however, in the framework of the single-particle interpretation of operator $\boldsymbol{J}^2$ containing an additional extension parameter μ (cf. after Eq. (39) below) and imposing the (essential) self-adjointness condition for this operator with Dirac's or Schwinger's string, he obtained the charge quantization rules (16) from the boundary conditions for its eigenfunctions that in fact make these strings invisible.

Let us examine the operator $\boldsymbol{J}$ in more detail. Though its initial expression is not a sum of angular momentum operators

$$\boldsymbol{J} = \boldsymbol{\mathcal{L}} - Q\mathbf{n} = \left(\mathbf{x}\times\left(\mathbf{p} - \frac{e}{c}\mathbf{A}_{\mathbf{u}}(\mathbf{x})\right)\right) - Q\mathbf{n}\,, \tag{32}$$

the use of the basis vectors of spherical and polar coordinate systems defined in Eqs. (6)–(9) and Schwinger's type of the vector potential of Eq. (11) recasts $\boldsymbol{J}$ into the following:

$$\boldsymbol{J} = i\hbar\left[\frac{\boldsymbol{\eta}_{(\beta)}}{\sin\beta}\frac{\partial}{\partial\alpha} - \boldsymbol{\eta}_{(\alpha)}\frac{\partial}{\partial\beta}\right] - \frac{\boldsymbol{\eta}_{(\rho)}}{\sin\beta}Q\,. \tag{33}$$

Note that the basis vectors used here are not fully mutually orthogonal. The main observation of this work, surprisingly still not explicitly mentioned in the literature (see Refs. 3, 4 and references therein) is that Cartesian components of this expression with Lipkin's[7] and Hurst's[9] substitutions both together:

$$-\,Q \leftrightarrow (\boldsymbol{J}\cdot\mathbf{n}) \leftrightarrow -i\hbar\frac{\partial}{\partial\gamma}\,, \tag{34}$$

exactly coincide, as it may be easily seen, with the standard expressions for Cartesian components of a total angular momentum operator of rotating rigid body — a top, in terms of its Euler angles α, β, γ as dynamical variables[10,11]:

$$J_1(\alpha,\beta,\gamma) = i\hbar\left[\cot\beta\cos\alpha\frac{\partial}{\partial\alpha} + \sin\alpha\frac{\partial}{\partial\beta} - \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right], \tag{35}$$

$$J_2(\alpha,\beta,\gamma) = i\hbar\left[\cot\beta\sin\alpha\frac{\partial}{\partial\alpha} - \cos\alpha\frac{\partial}{\partial\beta} - \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right], \tag{36}$$

$$J_3(\alpha,\beta,\gamma) = -i\hbar\frac{\partial}{\partial\alpha}\,, \tag{37}$$

what is in exact correspondence with the meaning of the value of $(\boldsymbol{J}\cdot\mathbf{n})$ as a projection of the total angular momentum operator onto the rotating axis $\mathbf{n}$. Keeping in mind the Eq. (33), may be the most explicit demonstration would be:

$$\boldsymbol{J}(\alpha,\beta,\gamma) = i\hbar\left[\frac{\boldsymbol{\eta}_{(\beta)}}{\sin\beta}\frac{\partial}{\partial\alpha} - \boldsymbol{\eta}_{(\alpha)}\frac{\partial}{\partial\beta} - \frac{\boldsymbol{\eta}_{(\rho)}}{\sin\beta}\frac{\partial}{\partial\gamma}\right]. \tag{38}$$

The expressions (35)–(38) have nothing to do with singularities of Schwinger's string (11), leading to the usual operator of total angular momentum square[10]: $\boldsymbol{J}^2 = J_1^2 + J_2^2 + J_3^2$, or:

$$\boldsymbol{J}^2(\alpha,\beta,\gamma) = (i\hbar)^2\left[\frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\left(\sin\beta\frac{\partial}{\partial\beta}\right)\right.$$
$$\left. + \frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} - 2\cos\beta\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\gamma} + \frac{\partial^2}{\partial\gamma^2}\right)\right]. \tag{39}$$

This expression was previously used for a charge–monopole system[2–4,9] only with fixed eigenvalues of the operators $J_3 \mapsto \hbar m$, and/or $(\boldsymbol{J}\cdot\mathbf{n}) \mapsto \hbar\mu$. For the last case Hurst[9] gives also the expressions for the Cartesian components of J_k (32) with Dirac's string (10) that however have nothing to do with Eq. (33) and the top angular momentum (35)–(38).

In the Cartesian basis of fixed coordinate system $\mathbf{e}_i$ the basis of the rotating coordinate system $\mathbf{f}_{(j)}$ tightly associating with the rotating top for $i, j = 1, 2, 3$, has the following form[10]:

$$\mathbf{f}_{(j)}(\alpha\beta\gamma) = \sum_{i=1}^{3} R_{ij}(\alpha\beta\gamma)\mathbf{e}_i\,, \qquad (\mathbf{e}_i\cdot\mathbf{e}_j) = (\mathbf{f}_{(i)}\cdot\mathbf{f}_{(j)}) = \delta_{ij}\,. \tag{40}$$

Conversion to this rotating system is realized by rotation matrix $\hat{R}(\alpha\beta\gamma)$ with matrices elements $R_{ij} = (\mathbf{e}_i\cdot\mathbf{f}_{(j)})$ given in Ref. 10 depending on the Euler angles: $0 \le \alpha < 2\pi$, $0 \le \beta \le \pi$, $0 \le \gamma < 2\pi$,

$$\hat{R}(\alpha\beta\gamma) = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma & -\cos\alpha\cos\beta\sin\gamma & \cos\alpha\sin\beta \\ -\sin\alpha\sin\gamma & -\sin\alpha\cos\gamma & \\ \sin\alpha\cos\beta\cos\gamma & -\sin\alpha\cos\beta\sin\gamma & \sin\alpha\sin\beta \\ +\cos\alpha\sin\gamma & +\cos\alpha\cos\gamma & \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix}. \tag{41}$$

Because for $l = 1, 2, 3$, the vectors $\mathbf{f}_{(l)}$ are vector operators with respect to rotation generated by operator $\boldsymbol{J}$ (38), the components of this differential operator in the rotating coordinate system are scalars $J_{(l)} \equiv \mathcal{P}_{(l)}$, defined as[10]:

$$\mathcal{P}_{(l)} = (\mathbf{f}_{(l)} \cdot \boldsymbol{J}) = \sum_{i=1}^{3} R_{il} J_i \,, \quad \text{where } [J_i, R_{il}] = 0 \,, \tag{42}$$

$$[J_i, R_{jl}] = [J_i, (\mathbf{f}_{(l)})_j] = i\hbar\epsilon_{ijk}(\mathbf{f}_{(l)})_k = i\hbar\epsilon_{ijk}R_{kl} \,, \tag{43}$$

$$\mathcal{P}_{(1)} = \mathcal{L}_{(1)} = i\hbar\left[-\cot\beta\cos\gamma\frac{\partial}{\partial\gamma} - \sin\gamma\frac{\partial}{\partial\beta} + \frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}\right], \tag{44}$$

$$\mathcal{P}_{(2)} = \mathcal{L}_{(2)} = i\hbar\left[\cot\beta\sin\gamma\frac{\partial}{\partial\gamma} - \cos\gamma\frac{\partial}{\partial\beta} - \frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}\right], \tag{45}$$

$$\mathcal{P}_{(3)} = (\boldsymbol{J} \cdot \mathbf{n}) = -i\hbar\frac{\partial}{\partial\gamma} \,, \qquad \mathcal{L}_{(3)} = (\mathbf{n} \cdot \boldsymbol{\mathcal{L}}) = 0 \,, \tag{46}$$

— since the vector $\mathbf{f}_{(3)} \equiv \mathbf{n}(\beta, \alpha)$ for all γ. These components obey the relations[10] (* — means complex conjugation):

$$[\mathcal{P}_{(i)}, \mathcal{P}_{(j)}] = -i\hbar\epsilon_{ijk}\mathcal{P}_{(k)} \,, \qquad [J_i, \mathcal{P}_{(j)}] = 0 \,, \tag{47}$$

$$\mathcal{P}_{(k)}(\alpha, \beta, \gamma) = J_k^*(-\gamma, -\beta, -\alpha) \,. \tag{48}$$

The operator (39) of total angular momentum square coincides for both coordinate systems.[10] Therefore, the angle-dependent part $\boldsymbol{\mathcal{L}}^2/r^2$ of the Hamiltonian (25) in fact represents the Hamiltonian of symmetric top[10,11] with an infinite moment of inertia about the third principal axis of inertia, $\mathbf{f}_{(3)} = \mathbf{n}$. The projection $\mathcal{P}_{(3)}$ onto this axis of the total angular momentum $\boldsymbol{J}$ then remains to be a constant $-Q$ and with $\boldsymbol{\mathcal{P}}^2 = \boldsymbol{J}^2$:

$$\boldsymbol{\mathcal{L}}^2 = \boldsymbol{J}^2 - Q^2 \leftrightarrow \boldsymbol{\mathcal{P}}^2 - \mathcal{P}_{(3)}^2 = \mathcal{P}_{(1)}^2 + \mathcal{P}_{(2)}^2 \,, \tag{49}$$

which can be considered as the main result of this work.

3. The Wave Function and the Scattering Amplitude

The above observations reveal a deep similarity between the rotation wave functions of the usual spinless diatomic molecule with taking into account the total orbital momentum of its electronic shell,[11] and the rotation wave functions of the effective "molecule" composed by the electric charge and magnetic monopole (or by the two dyons) with taking into account the angular momentum of their common EMF. Thus for the states with the fixed total angular momentum for both these "molecules", $\boldsymbol{J}^2 \mapsto \hbar^2 j(j+1)$, the states of the usual molecule with conserving projection of its averaged electronic shell orbital momentum onto the rotating molecule's axis $\mathbf{n}$: $(\bar{\mathbf{L}}_e \cdot \mathbf{n}) = (\boldsymbol{J} \cdot \mathbf{n}) \mapsto \hbar\lambda$, with the obvious condition[11] $j \geq |\lambda|$, are in direct correspondence with the states of the charge–monopole "molecule" with a certain conserved projection of the angular momentum of their common EMF[3,4] onto the rotating "molecule's" axis $\mathbf{n}$: $(\mathbf{S}\cdot\mathbf{n}) = (\boldsymbol{J}\cdot\mathbf{n}) \mapsto \hbar\mu = -Q$, what eventually gives the one and the same conditions for the one and the same eigenfunctions of symmetric quantum top for the "molecules" of both types, with the replacement[2,3,11]: $\lambda \leftrightarrow \mu$. For matrix representation J_k of angular momentum operators, with $J_k \mapsto \hbar \mathrm{J}_k$, these well-known Wigner's D-functions[10] appear from:

$$[\mathrm{J}_i, \mathrm{J}_j] = i\epsilon_{ijk}\mathrm{J}_k\,, \qquad \mathcal{U}(\alpha,\beta,\gamma) = e^{-i\alpha \mathrm{J}_3} e^{-i\beta \mathrm{J}_2} e^{-i\gamma \mathrm{J}_3}\,, \tag{50}$$

$$\frac{1}{i^j}\sqrt{\frac{8\pi^2}{2j+1}}\langle \alpha\beta\gamma|jm\rangle_\mu = \langle j\mu|\mathcal{U}^{-1}(\alpha,\beta,\gamma)|jm\rangle \tag{51}$$

$$= D^{(j)*}_{m,\mu}(\alpha,\beta,\gamma) = e^{i\mu\gamma} d^j_{m,\mu}(\cos\beta) e^{im\alpha}\,, \tag{52}$$

as the common eigenfunctions of the operators $\boldsymbol{J}^2$, J_3 and $(\boldsymbol{J}\cdot\mathbf{n}(\beta,\alpha)) = \mathcal{P}_{(3)}$ with the eigenvalues $\hbar^2 j(j+1)$, $\hbar m$ and $\hbar\mu$ respectively, for which $-j \le m,\ \mu \le j$. When $\gamma = -\alpha$, these eigenfunctions are reduced to[10,11]:

$$\mathcal{U}^{-1}(\alpha,\beta,-\alpha) = e^{-i\alpha \mathrm{J}_3} e^{i\beta \mathrm{J}_2} e^{i\alpha \mathrm{J}_3} = \exp\{i\beta(\mathbf{J}\cdot\boldsymbol{\eta}_{(\alpha)})\}\,, \tag{53}$$

$$\frac{1}{i^j}\sqrt{\frac{4\pi}{2j+1}}\langle \mathbf{n}(\beta,\alpha)|jm\rangle_\mu = \langle j\mu|\mathcal{U}^{-1}(\alpha,\beta,-\alpha)|jm\rangle\,, \tag{54}$$

giving:
$$\langle \alpha\beta\gamma|jm\rangle_\mu = \frac{e^{i\mu(\gamma+\alpha)}}{\sqrt{2\pi}}\langle \mathbf{n}(\beta,\alpha)|jm\rangle_\mu\,. \tag{55}$$

Thus, for the gauge $\mathbf{u} = \mathbf{e}_z$ Schwinger's string is "dissolved" in the total angular momentum operator for "charge + monopole + common EMF" if the latter is considered as a total angular momentum operator of some effective extended quantum object with the properties of the symmetric top. So, the above eigenvalues j, m, μ of the mutually commutative (differential) operators $\boldsymbol{J}^2$, J_3 and $\mathcal{P}_{(3)}$, can be integer as well as half integer. Indeed, unlike the usual diatomic molecules, the additional common EMF of charge and monopole should not induce here only purely integer orbital momentum, whereas the disappearance of the string makes irrelevant Schwinger's narrowing[6] onto the even N. Note that the notion of extended (impenetrable rigid) body in quantum mechanics admits both integer and half integer values of its angular momentum.[10]

When the charge falls along the z axis from $z = -\infty$, one has $\mathbf{n} = -\mathbf{e}_z$ and for the eigenvalue $\hbar m$ of the operator $J_3 = (\boldsymbol{J}\cdot\mathbf{e}_z) \mapsto -(\boldsymbol{J}\cdot\mathbf{n})$ obtains[2,3] $m = -\mu$. Thus, the exact scattering wave function $\psi^{(+)}_{\mathbf{k}}(\mathbf{x})$ and the full scattering amplitude $\mathcal{F}(k^2, \cos\beta)$ are connected by the relation[1,3]:

$$\psi^{(+)}_{\mathbf{k}}(\mathbf{x}) = e^{-i\pi\mu}\sum_{j=|\mu|}^{\infty}(2j+1)e^{i\pi j}e^{-i\pi\ell/2}\,\mathrm{j}_\ell(kr) D^{(j)*}_{-\mu,\mu}(\alpha,\beta,-\alpha)\,, \tag{56}$$

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{x}) \xrightarrow[r\to\infty]{} e^{-2i\mu\alpha}\left[e^{i(\mathbf{k}\cdot\mathbf{x})} + \mathcal{F}(k^2,\cos\beta)\frac{e^{ikr}}{r}\right], \tag{57}$$

$$2ik\mathcal{F}(k^2,\cos\beta) = e^{-i\pi\mu}\sum_{j=|\mu|}^{\infty}(2j+1)e^{-i\pi(\ell-j)}d^{j}_{-\mu,\mu}(\cos\beta)\,, \tag{58}$$

what follows[3] from asymptotic behavior of the Bessel function $\mathrm{j}_\ell(kr)$:

$$\mathrm{j}_\ell(kr) \xrightarrow[r\to\infty]{} \frac{1}{kr}\sin\left(kr - \frac{\pi\ell}{2}\right), \qquad \ell + \frac{1}{2} = \left[\left(j+\frac{1}{2}\right)^2 - \mu^2\right]^{1/2}. \tag{59}$$

When in Eq. (56) the eigenfunctions of a symmetric top of Eq. (52) is used, the multiplier $e^{-2i\mu\alpha}$ that deforms also the falling plane wave[2,3] in expression (57), is replaced by $e^{i\mu(\gamma-\alpha)}$. However, for the fixed μ the dependence on angle γ coming here from Eq. (55) gives only a common phase as well as for the case of the usual diatomic molecule.[11] Therefore, this multiplier has no physical meaning and can not change the one-particle interpretation of the scattering wave function (56) and scattering amplitude (58), because the vector $\mathbf{n}$ in Eq. (6) depends on Euler angles α, β only, where β becomes a scattering angle.

4. Conclusion

The inability to assign irremovable additional angular momentum (17) of common charge–monopole EMF to any of these particles indicates incompleteness of single-particle interpretation[9] of the rotation symmetry for this system and lack of single-particle interpretation for the total angular momentum operator and its eigenfunctions. We showed that Schwinger's symmetric vector potential[6] (1), (11) directly leads to the more natural interpretation of the rotation symmetry for this system, as a symmetry of extended object with the properties of a symmetric quantum top with the infinite moment of inertia about the third principal axis of inertia. The whole system behaves with respect to rotations similarly to diatomic molecule with taking into account the total angular momentum of its electronic shell. Adjacent results with different interpretation via "spinning-isospinning top" were obtained in Ref. 12 starting from purely classical consideration of the charge-monopole system.

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