HIGH-ENERGY COSMIC-RAY MUONS UNDER THICK LAYERS OF MATTER

I. A Method to Solve the Transport Equation[*]

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An effective analytical method for calculating energy spectra of cosmic-ray muons at large depths of homogeneous media is developed. The method allows to include an arbitrary (decreasing) muon spectrum at the medium boundary and the energy dependence of both discrete (radiative and photonuclear) and continuous (ionization) muon energy losses, with resonable requirements for the high-energy behavior of the initial spectrum and differential cross sections of the muon-matter interactions.

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I. INTRODUCTION

Cosmic-ray (CR) muons originate from the decay of unstable hadrons produced by the interactions of cosmic-ray primaries and secondaries with nuclei of the earth's atmosphere. Therefore the flux of CR muons contains information on primary cosmic rays (energy spectrum, composition, anisotropy) as well as on some properties of particle interactions at high and super-high energies.

During the last years the experimental investigations of CR muons with large low-background detectors for penetrating particles have expanded rapidly in a number of underground laboratories, in addition to the direct measurements in the atmosphere and at ground level. Side by side with the traditional range of problems of cosmic ray physics some additional aspects arise within the framework of investigations with the new facilities. Thus, for example, the flux of CR muons is used for calibration of the detectors and, at the same time, it is an important source of background events for the majority of underground experiments, especially in neutrino astronomy and astrophysics. Detailed study of this background is very important for further progress in astroparticle physics.

Projects for the deep-underwater Čerenkov and acoustic detection of high-energy muons and neutrinos have been discussed for a long time. The ultimate aim of these projects is to build detectors of volume $10^7 - 10^9$ m^3 or even larger [1,2] which could be used, in particular, to detect muons of energy up to 10^3 TeV, in order to study the CR muon flux at energies 2 to 3 orders of magnitude higher than those accessible in the present experiments. It should be particularly emphasized that detectors of so enormous volume can be used to accurately determine the energy, E, of individual muons passing through the apparatus if E exceeds a few TeV [1].

In the near future the initial stage of the underwater muon and neutrino telescopes DUMAND II in the ocean off the Hawaii island [3] and NT-200 in the Siberian lake Baikal

[4] will be commissioned. Besides, several new programs have been proposed, such as a deep sea neutrino detector NESTOR in the southwest corner of the Peloponnisos [5] and the project AMANDA for a large scale muon and neutrino detector in deep ice at the South Pole [6] (see also Ref. [7] for a short summary on next generation detectors). Precision calculations of different characteristics of the CR muon flux after propagation through thick water layers are an imperative element for the successful realization of these projects.

The transport of high-energy CR muons through dense media has been the subject of theoretical investigations over many years with the use of analytical [8–16], numerical [17,18], and Monte Carlo [19–21] methods (see also Refs. [22,23] for a review of the early literature). In the majority of the papers listed the depth-intensity relation (DIR) was studied. However, for future experiments with large-scale underwater neutrino telescopes a detailed knowledge of the energy spectra of muons at very large depths (large zenith angles) will be required in addition to the total (oblique) intensities. Some results of calculations of the muon energy spectra at large depths of matter were presented in Refs. [9,20,21] and in our previous papers [14,15], but the increasing requirements on accuracy of the calculations stimulate us to continue the investigation of the problem.

The main difficulty in the calculation of muon intensity and spectrum at large depths consists in the fact that an ultrarelativistic muon of energy E above ~ 100 GeV can, with comparable probabilities, lose in a single event either a very small energy $\Delta E \ll E$ or an energy $\Delta E \sim E$ with generation of a large electromagnetic or hadronic shower due to radiative or photonuclear interaction with matter. These fluctuations of the energy loss lead to a pronounced range straggling. Considering that the rate of radiative and photonuclear losses increases with energy, the fluctuation effect grows with energy and depth. An important consequence of this effect is the impossibility to define a threshold energy for a muon reaching a given depth. This fact presents a severe problem when reconstructing the surface

(sea-level) muon spectrum from a measured DIR (see, *e.g.*, Ref. [24])[1].

Available exact analytical methods of cascade theory [11] require that the initial muon spectrum at the boundary of the medium be a power-law and that the differential cross sections for muon-matter interactions depend only on the fraction of energy lost by the muon, $v = \Delta E/E$, but not on the muon energy, E, itself ("scaling"). It is also assumed that the rate of the continuous (ionization) energy loss is constant. We shall call this set of assumptions the SPS model (Scaling + Power-law Spectrum). Such model have been considered after Rozental' and Strel'tsov [28] in Refs. [8–10]. Zatsepin and Mikhalchi [8] have suggested a very simple approximate solution to the muon transport equation (TE). Their approach has been generalized to a quasi-power initial spectrum [9]. The exact solution for DIR within the framework of the SPS model has been obtained by Nishimura [10] (see also Refs. [11–13]) with the use of the technique of integral transformations. Both these approaches have been employed with some modifications in numerous works (see, *e.g.*, Refs. [16,21]).

However, as is generally known, the assumptions of the SPS model hold roughly only at very high muon energies (above 1–10 TeV), and so the calculations based on the SPS solution should be corrected by one or another way. Despite the relatively weak energy dependence of the differential cross sections as well as the closeness of the real sea-level muon spectrum to a power-law form within wide energy intervals, these corrections prove to be very large and they increase with depth. The point is that the muon energy spectrum under thick

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[1]In this connection the problem of prompt muons which appear in the atmosphere due to decays of charmed hadrons should be mentioned [25,26]. The data of the current underground experiments, those from European detectors (NUSEX, Frejus, MACRO), on the one hand, and those from Baksan and KGF, on the other hand, contradict each other (see Ref. [26]). A certain part of these disagreements can be attributed to an inaccuracy in the computation of the fluctuation effect (see also Ref. [24] and a discussion in Ref. [27]).

layers of matter depends exponentially on integrals of the differential cross sections with a weight which depends on the initial spectrum. To our knowledge there is not any proper and consistent way to calculate these corrections at large depths for the time being.

In the present study we discuss a comparatively simple and universal method for the calculation of differential energy spectra as well as other important characteristics of CR muons at arbitrary depth, which allows us to avoid from the start the simplifying assumptions about the scale invariance of the cross sections and the (quasi) power-law incident muon spectrum. The solution to the TE is constructed by iterations, starting from an initial approximation with the correct high-energy asymptotic behavior. In the range of applicability of the initial approximation (sufficiently high energies) it becomes feasible to introduce an (effective) analog of the threshold energy at the boundary which is very useful in many respects. One of the advantages of the computer realization of our approach (in comparison with the direct Monte Carlo simulation or a purely numerical technique) is its high performance which allows to carry out verifications of various hypotheses on the primary CR spectrum and composition, charm production models, models of the photonuclear interaction, *etc* with good precision and in a negligible CPU time. This enables to estimate the sea-level muon spectrum using the data of underground/underwater measurements by exhaustion, avoiding to solve the much more difficult inverse scattering problem.

It should be noted that the method under consideration is a development of our previous studies [14,15].

The organization of this paper is as follows. In Sec. II we give some preliminaries and notations. We present also a very short review on some features of the differential cross sections for muon-matter interactions at high energies which will be needed later on. In Sec. III the solution to the TE in a continuous loss approximation is discussed briefly for the methodological goals. In Sec. IV we consider the exact solution to the TE within the SPS model; for the present purpose (to study the asymptotic behavior of the TE solution at high energies) the simplest expression in the form of a series in powers of $1/E$ will be quite enough. In Sec. V we derive an approximate solution to the TE in the general case; essential properties of the solution are discussed in some detail and illustrated by the SPS model. The iteration algorithm for calculating corrections to the approximate solution is described in Sec. VI and the convergency of the algorithm is examined. Finally, in Sec. VII we summarize the results and some perspectives for applications of the method.

II. PRELIMINARIES

A. Transport equation

The propagation of relativistic muons through a homogeneous medium is described by the one-dimensional transport equation (TE)

$$\frac{\partial}{\partial x}D(E,x) - \frac{\partial}{\partial E}[\beta_i(E)D(E,x)] = \langle D(E,x)\rangle \qquad (2.1)$$

with the boundary condition

$$D(E,0) = D_0(E) . \qquad (2.2)$$

Here $D(E,x)$ is the differential energy spectrum of muons at depth x in the medium. In the general case

$$x = \sec\vartheta \int_0^z \rho(z')dz' ,$$

where $\rho(z)$ is the density of the medium at distance z from the boundary, and ϑ is the angle of incidence measured from the normal to the boundary (zenith angle). The function $\beta_i(E) = -(dE/dx)_{\mathrm{ion}}$ is the rate of the ionization energy losses which, as ever, are assumed to be continuous. The symbol $\langle D\rangle$ denotes a functional describing the "discrete" muon energy loss resulting from radiative and photonuclear processes:

$$\langle D(E,x)\rangle = \sum_k \langle D(E,x)\rangle_k , \qquad (2.3)$$

$$\langle D(E,x)\rangle_k = \left\langle \frac{N_0}{A} \int \frac{d\sigma_k^{Z,A}(E_1,E)}{dE} D(E_1,x)dE_1 \right\rangle_{Z,A}$$
$$- \left\langle \frac{N_0}{A} \int \frac{d\sigma_k^{Z,A}(E,E_2)}{dE_2} D(E,x)dE_2 \right\rangle_{Z,A} . \qquad (2.4)$$

Here $d\sigma_k^{Z,A}(E_1,E_2)/dE_2$ is the differential cross section for a muon interaction of type k:

$(k = n)$, and E_1 (E_2) is the initial (final) muon energy; N_0 is the Avogadro number; Z and A are the atomic number and atomic weight of the target nucleus. The brackets $\langle \ldots \rangle_{Z,A}$ indicate an averaging over Z and A. Integrations in Eq. (2.4) are performed between the limits allowed by the k-type process kinematics:

$$E_{1,\min}^k(E) \leq E_1 \leq E_{1,\max}^k(E) \,, \qquad E_{2,\min}^k(E) \leq E_2 \leq E_{2,\max}^k(E) \,.$$

Equation (2.1) does not take into account the muon finite lifetime, which is permissible for ultrarelativistic energies and/or for dense enough media[2]. Moreover, in the equation (2.1) (valid within the "straight-forward" scattering approximation) multiple Coulomb scattering and the angular deflection due to inelastic scattering have been ignored. This approximation is not so inoffensive but an examination of the problem does not enter the scope of the present work.

A way to include the fluctuation effect due to knock-on electron production by muons will be considered later on. An estimation of this effect for DIR has been made by Nishimura [12]. According to Ref. [12] the effect leads to an increase of DIR at all depths by approximately 3% (in the special case of an initial spectrum $D_0(E) \propto E^{-4}$). A reliable analytical method for describing the ionization straggling of relativistic muons with incident energies below ~ 100 GeV in thick absorbers has been suggested recently by Striganov [29], but processes others than ionization were not taken into account.

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[2]The average decay range of a muon of energy E is given by

$$\lambda_d(E) = \tau_\mu p \rho/(m_\mu c) \simeq 6.23 \times 10^5 \text{ g/cm}^2 \left(\frac{\rho}{1 \text{ g/cm}^3} \right) \left(\frac{p}{1 \text{ GeV/c}} \right) \,,$$

where m_μ, τ_μ, and $p = \sqrt{(E/c)^2 - (m_\mu c)^2}$ are the muon mass, lifetime, and momentum, respectively. Clearly $\lambda_d(E)$ is much longer than the muon ionization range $\lambda_i(E)$ [30] in a dense medium, so the muon decay effect is totally unessential in all instances of interest.

Let us introduce the macroscopic cross sections Σ_k by the definition

$$\Sigma_k(v, E) = \left\langle \frac{N_0}{A} \frac{d\sigma_k^{Z,A}(v, E)}{dv} \right\rangle_{Z,A} , \tag{2.5}$$

where

$$\frac{d\sigma_k^{Z,A}(v, E)}{dv} = \left| \frac{E d\sigma_k^{Z,A}(E, E')}{dE'} \right|_{E'=(1-v)E} ,$$

and v is the fraction of energy lost. With help of Eq. (2.5) we rewrite Eq. (2.4) in the more convenient form:

$$\langle D(E, x) \rangle_k = \int_0^1 [(1-v)^{-1} \Phi_k(v, E_v) D(E_v, x) - \Phi_k(v, E) D(E, x)] dv . \tag{2.6}$$

Here and below

$$\Phi_k(v, E) = \theta(v_{\max}^k(E) - v) \theta(v - v_{\min}^k(E)) \Sigma_k(v, E) , \tag{2.7}$$

$\theta(x)$ is the usual step function, $v_{\min}^k(E)$ and $v_{\max}^k(E)$ are the extreme values of v for the k-type process, $E_v \equiv E/(1-v)$, and the function $\Phi_k(v, E_v)$ is defined by Eq. (2.7) with the substitution $E \Rightarrow E_v$.

At ultrarelativistic energies ($E \gg m_\mu c^2$, specifically at E above ≈ 10 GeV), we have with sufficient accuracy that

$$v_{\min}^k(E) = 0 , \quad v_{\max}^k(E) = 1 , \tag{2.8}$$

and hence $\Phi_k(v, E) = \Sigma_k(v, E)$. Moreover, this may formally be extended to all energies considering that radiative and photonuclear losses are inessential to an accuracy of about 1% at energies under 10 GeV for all media of interest in cosmic ray physics [30,31], and only ionization losses are important. Nevertheless, in the following we shall use approximation (2.8) only for asymptotic estimations, but we shall assume, if necessary, that $v_{\min}^k(E) \ll 1$ and $1 - v_{\max}^k(E) \ll 1$ for $k = p, b, n$ in the energy region covered.

B. Some features of muon-matter interactions at high energies

A detailed description of the cross sections $d\sigma_k^{Z,A}(v,E)/dv$ used in our calculations will be presented in a separate publication. For a short review, see Ref. [31]. To provide an inside into the properties of the radiative processes we have presented in the Appendix a very simple parameterization of the v-dependencies of the (normalized) cross sections suggested by van Ginneken [32]. As one can see from the Appendix, strong energy loss fluctuations are more probable in bremsstrahlung. The direct pair production cross section goes roughly as $1/v^2$ to $1/v^3$ over most of the range ($v > 0.002$). Usually these losses are treated as continuous. Nevertheless, as it follows from our estimations, the fluctuation effect related to pair production is not exactly negligible and it can prove essential at large depths. Hence we will be considering the pair production contribution as discrete, together with bremsstrahlung.

To this must be added that in the limit of complete screening, i.e. for

$$\gamma_Z(v,E) \equiv \frac{200 q_{\min}}{m_e Z^{1/3}} \simeq \left(\frac{11}{Z}\right)^{1/3} \left(\frac{1\,\text{TeV}}{E}\right) \frac{v}{1-v} \ll 1$$

(where γ_Z is the degree of screening and $q_{\min} \simeq m_\mu^2 v/[2E(1-v)]$ is the minimum momentum transfer), the radiative cross sections are functions of the variable v only (scaling). However for values of v which are not too small (namely, at $1-v \ll 1$) complete screening occurs only at very high energies, $E \sim 10$ TeV. At lower energies the cross sections grow logarithmically with E.

Unfortunately there is no simple parameterization for the differential cross section of the inelastic muon scattering on a nucleus $d\sigma_n^{Z,A}/dv$. Moreover, both v- and (especially) E-behavior of the cross section are very model dependent.

According to the vector-meson-dominance hypothesis $d\sigma_n^{Z,A}/dv$ is expressed in terms of the total cross section for virtual photon absorption by nucleons and nuclei. A generalized

tions in the diffraction region (low 4-momentum transfers, Q^2, and large photon energies, ν): growth with energy of the cross section for nucleon photoabsorption and shadowing effects in nuclear photoabsorption. An approximate expression for $d\sigma_n^{Z,A}/dv$ has been evaluated in the framework of the GVDM by Bezrukov and Bugaev [33]:

$$d\sigma_n^{Z,A}(v,E)/dv \propto \sigma_{\gamma N}(\nu) F_n(v,\nu)/v \ .$$

Here $\sigma_{\gamma N}(\nu)$ is the total cross section for absorption of a real photon of energy $\nu = vE$ by a nucleon. In agreement with accelerator and cosmic-ray experiments [34,35] $\sigma_{\gamma N}(\nu)$ grows slowly above $\nu \sim 50$ GeV and can be represented approximately as

$$\sigma_{\gamma N}(\nu) \simeq \left[114.3 + 1.647 \ln^2 \left(\frac{\nu}{47\,\mathrm{GeV}} \right) \right] \ \mu\mathrm{b} \ .$$

The growth of $\sigma_{\gamma N}(\nu)$ causes $d\sigma_n^{Z,A}/dv$ to depend on the muon energy, E. The function $F_n(v,\nu)$ decreases slowly with increasing ν, gradually compensating the energy dependence of $\sigma_{\gamma N}$ (a manifestation of the shadowing effect of nucleons inside a target nucleus). Nevertheless, the logarithmic growth of $d\sigma_n^{Z,A}/dv$ quantitatively remains up to $E \sim 10$ TeV and possibly in the asymptotics. The v-dependence of $F_n(v,\nu)$ is rather complicated; for v over ~ 0.1 it falls off roughly as $\ln v$ with increasing v, thus the fluctuation effect due to this process is comparatively large.

It should be mentioned that the absence of any unitarity constraint allows a very rapid (in comparison with the GVDM prediction) increase with energy of the total photoproduction cross section as a result of the gluonic structure of the high-energy photon ("minijet production mechanism") [36]. Although available cosmic-ray data obtained with underground detectors [34] (for ν up to ~ 10 TeV) and with EAS arrays [35] (ν up to $10^3 - 10^4$ TeV !) do not support this possibility, and what is more, a recent study [37] has shown that the calculations of Ref. [36] strongly overestimate the minijet production contribution at $\nu > 10^3$ TeV, a significant increase of the photoproduction cross section is still anticipated

at ultra-high energies. Thus the photonuclear interaction is one of the interesting objects for study in future experiments with large underground and underwater detectors.

III. CONTINUOUS LOSS APPROXIMATION

Let us at first consider the so-called continuous loss (CL) approximation which is often-used for estimations of the CR muon intensity and spectrum under thin enough layers of matter (see, *e.g.*, Refs. [23,24] and [38]). It can be obtained from Eq. (2.1) by a formal expansion of the integrand of expression (2.6) in powers of E_v at $E_v = E$, to an accuracy of $O(v)$. As a result the functional (2.3) becomes

$$\langle D(E,x) \rangle = \sum_k \int_0^1 (1 + E\frac{\partial}{\partial E})\Phi_k(v,E)D(E,x)v dv .$$

Let us define

$$b_k(E) = \int_0^1 \Phi_k(v,E)v dv = \int_{v_{\min}^k(E)}^{v_{\max}^k(E)} \Sigma_k(v,E)v dv .$$

Clearly $b_k(E)$ is the relative partial rate of average energy loss due to the k-type process, and

$$\beta(E) = \beta_i(E) + E\sum_k b_k(E) = -(dE/dx)_{\text{tot}}$$

is the total rate of energy loss. Thus Eq. (2.1) in the CL approximation takes the form

$$\frac{\partial}{\partial x}\overline{D}(E,x) = \frac{\partial}{\partial E}[\beta(E)\overline{D}(E,x)] . \tag{3.1}$$

Here and below $\overline{D}(E,x)$ stands for the differential muon spectrum in the CL approximation. Similarly $\overline{I}(E,x)$ and $\overline{J}(x)$ will stand for integral spectrum and DIR, respectively.

The solution to Eq. (3.1) with boundary condition (2.2) is

$$\overline{D}(E,x) = D_0(\mathcal{E}(E,x))\frac{\beta(\mathcal{E}(E,x))}{\partial(E)} , \tag{3.2}$$

where $\mathcal{E}(E,x)$ is the (only) root of the equation $\lambda(\mathcal{E},E) = x$, and

$$\lambda(E_1, E_2) = \int_{E_2}^{E_1} \frac{dE}{\beta(E)}$$

is the average range of a muon with initial energy E_1 and final energy E_2. In other words, $\mathcal{E}(E,x)$ is the energy which a muon must have at the boundary of the medium in order to reach depth x with energy E. It is easily seen that $\mathcal{E}(E,x)$ is a monotonically increasing function of variables E and x, and the following identities are valid for any $x' \leq x$ and $E' \geq E$:

$$\mathcal{E}(\mathcal{E}(E,x'), x - x') = \mathcal{E}(E', x - \lambda(E', E)) = \mathcal{E}(E,x) .$$

It is clear also that $\mathcal{E}(E,0) = E$.

From Eq. (3.2) a very nice expression for the integral muon spectrum at depth x can be obtained. Let $I_0(E) \equiv I(E,0)$ be the integral spectrum at the boundary, then

$$\overline{I}(E,x) = \int_E^\infty \overline{D}(E',x)dE' = \int_{\mathcal{E}(E,x)}^\infty D_0(E')dE' = I_0(\mathcal{E}(E,x)) . \tag{3.3}$$

According to Eq. (3.3) the expression for DIR, $J(x)$, can be written as

$$\overline{J}(x) = I_0(\mathcal{E}(E_t, x)) , \tag{3.4}$$

where E_t is some detection threshold. It can be argued that the value $\overline{J}(x)$ is practically independent of E_t at large depths when E_t is sufficiently low (really, when $E_t \ll 1$ TeV).

In spite of the simplicity and physical transparency of the CL approximation, its range of application is fairly restricted. The inadequacy of this approximation is obvious from the following simple example. Let the initial spectrum, $D_0(E)$, have a breakoff at some energy $E_{\max}$, i.e. $D_0(E) = 0$ at $E > E_{\max}$. Then, in accordance with Eqs. (3.2) and (3.3), $\overline{D}(E,x) = 0$ and $\overline{I}(E,x) = 0$ at $x > \lambda(E_{\max}, E)$. It is incorrect, of course, at least when $\lambda(E_{\max}, E) < \lambda_d(E)$. We will demonstrate below, within a simple model, that the CL

IV. ASYMPTOTIC BEHAVIOR (SPS MODEL)

We consider here the SPS model mentioned in Sec. I. Let us assume that the functions $\Phi_k(v, E)$ and the ionization loss rate, $\beta_i(E)$, are energy independent,

$$\Phi_k = \Phi_k(v) , \qquad \beta_i \equiv a = const , \tag{4.1}$$

and the initial spectrum is a power function of energy,

$$D_0(E) = D_0^\gamma(E) = CE^{-(\gamma+1)} . \tag{4.2}$$

Moreover muon energies are assumed to be high enough so that conditions (2.8) is fulfilled.

In the SPS model the rate of the average energy loss is simply $a + bE$, where $b = b_p + b_b + b_n$ is a constant[3], and, therefore, the differential and integral muon spectra in the CL approximation are described by

$$\overline{D}_{\mathrm{SPS}}(E, x) = D_0^\gamma(E)e^{-\gamma bx} \left[1 + \frac{a}{bE}(1 - e^{-bx})\right]^{-(\gamma+1)} , \tag{4.3}$$

$$\overline{I}_{\mathrm{SPS}}(E, x) = I_0^\gamma(E)e^{-\gamma bx} \left[1 + \frac{a}{bE}(1 - e^{-bx})\right]^{-\gamma} , \tag{4.4}$$

where $I_0^\gamma(E) = \gamma^{-1}CE^{-\gamma}$ is the initial integral spectrum.

A characteristic property of the SPS model in the CL approximation is a flat (energy independent) spectrum (both differential and integral) for $E \ll W \equiv a/b \sim 1$ TeV at sufficiently large depths ($x \gg 1/b$),

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[3]In reality the quantities b_k ($k = p, b, n$) and β_i grow with energy logarithmically (or as a power of logarithm) up to $E \sim 10$ TeV (see Refs. [30,31] and [39]). But, as we have noted in Sec. II B, it is not inconceivable, strictly speaking, that the growth of the relative rate of the average photonuclear loss extends at $E \gg 10$ TeV if a dramatic increase of the total photoproduction cross section with

$$\overline{D}_{\mathrm{SPS}}(E,x) \simeq D_0^\gamma(W)e^{-\gamma bx} , \quad \overline{I}_{\mathrm{SPS}}(E,x) \simeq I_0^\gamma(W)e^{-\gamma bx} ,$$

and recovery of its original form,

$$\overline{D}_{\mathrm{SPS}}(E,x) \simeq D_0^\gamma(E)e^{-\gamma bx} , \quad \overline{I}_{\mathrm{SPS}}(E,x) \simeq I_0^\gamma(E)e^{-\gamma bx} ,$$

for $E \gg W$ at all depths. According to Eq. (3.4), DIR takes the form

$$\overline{J}_{\mathrm{SPS}}(x) = I_0^\gamma(W(e^{bx} - 1)) ,$$

independently of the threshold energy E_t if $E_t \ll W(1 - e^{-bx})$.

Let us consider now the exact solution to Eq. (2.1) within the framework of the SPS model. Denote

$$b_{\gamma+n} = \int_0^1 \Phi(v)[1 - (1-v)^{\gamma+n}]dv , \quad n = 0, 1, \ldots ,$$

and

$$\varrho_\gamma = b_{\gamma+1} - b_\gamma = \int_0^1 \Phi(v)(1-v)^\gamma v \, dv ,$$

with $\Phi(v) = \Phi_p(v) + \Phi_b(v) + \Phi_n(v)$. We shall seek the solution as a series in powers of the dimensionless parameter $\xi = a/(\varrho_\gamma E)$:

$$D_{\mathrm{SPS}}(E,x) = D_0^\gamma(E)e^{-b_\gamma x} \sum_{n=0}^\infty \frac{(\gamma+1)_n}{n!} f_n(x)(-\xi)^n \qquad (4.5)$$

(here $(\ldots)_n$ is the Pochhammer symbol). Substituting Eq. (4.5) into Eq. (2.1), we find that the coefficient functions $f_n(x)$ satisfy the following recurrence formula:

$$f_n'(x) + (b_{\gamma+n} - b_\gamma)f_n(x) = n\varrho_\gamma f_{n-1}(x) , \qquad f_n(0) = \delta_{n0} . \qquad (4.6)$$

Integration of Eq. (4.6) yields

$$f_n(x) = \delta_{n0} + n\varrho_\gamma \int_0^x \exp[-(b_{\gamma+n} - b_\gamma)(x - x')]f_{n-1}(x')dx' . \qquad (4.7)$$

In particular, for $n = 0$ and 1 we have from Eq. (4.7) $f_0(x) = 1$ and $f_1(x) = 1 - e^{-\varrho_\gamma x}$.

By induction, and using the fact that $b_{\gamma+n} - b_\gamma < n\varrho_\gamma$ at $n \geq 1$, one can easily verify that

$$[f_1(x)]^n \leq f_n(x) \leq (\varrho_\gamma x)^n$$

for all values of x. Therefore the series (4.5) is absolutely and uniformly convergent under the condition

$$\zeta \equiv (\varrho_\gamma x)\xi = \frac{ax}{E} \leq 1 \, , \qquad (4.8)$$

but it is divergent when $\xi f_1(x) > 1$.

It can be shown that the obtained solution reduces to the solution in the CL approximation (4.3) if one sets formally $b_{\gamma+n} = (\gamma + n)b$, for $n \geq 0$. A rough fulfillment of these equalities at not too large values of n, which is a consequence of a quick growth of the electrodynamic cross sections at $v \ll 1$ (see the Appendix), serves as the basis for the applicability of the CL approximation. It is obvious, however, that for any $n \geq 0$ and $\gamma > 1$ the exact inequalities $b_{\gamma+n} < (\gamma + n)b$ take place, which are satisfied (irrespective of the behavior of the function $\Phi(v)$), in so far as $(1 - v)^t > 1 - tv$, at any $t > 1$ and $0 < v \leq 1$.

Thus the ratio

$$r(E, x) = D_{\mathrm{SPS}}(E, x)/\overline{D}_{\mathrm{SPS}}(E, x) \, ,$$

which is a measure of the fluctuation effect, increases with depth as $\exp[(\gamma b - b_\gamma)x]$ at $\xi \ll 1$. In other words the CL approximation underestimates the muon intensity at high energies. The magnitude of the effect depends critically on the slope of the initial spectrum (the remainder $\gamma b - b_\gamma$ quickly increases with γ) and it can be very large. To cite a single example, $r(E, x)$ is about 10 at $E = 10$ TeV and $x = 10$ km of water equivalent (for standard rock), in the case of the vertical spectrum of conventional CR muons from the decay of π

and K mesons [14]. It should be noted at the same time that the ratio $r(E, x)$ does not necessarily exceed unit at all energies.

The model under consideration shows that it is impossible to take into account the fluctuation effect on muon spectra at large depths as a correction to the CL approximation and a reliable method is required. Clearly the exact solution (4.5) by itself is unsuitable for calculations at fairly low energies and/or at large depths; it cannot be used, in particular, to compute DIR. At the same time the SPS model suggests a starting point for the required method: we may, using an *ansatz* which has the correct asymptotic behavior at high energies, construct the solution for the TE applying an appropriate iteration procedure. In the next sections we will consider this approach to the problem.

It will be convenient to specify the asymptotic behavior of the cross sections and initial spectrum at high energies as in the SPS model, that is to demand the fulfilment of equalities (4.1) and (4.2) at energies $E \gg E_{as}$, where E_{as} (a conventional bound of the asymptotic regime) is a sufficiently large quantity. Then the SPS model will serve as a base for asymptotic estimations. It will be recalled that the asymptotic form of the photonuclear cross section contribution $\Phi_n(v, E)$ is actually unknown as well as, strictly speaking, the high-energy behavior of the initial muon spectrum, $D_0(E)$. Nevertheless, the condition imposed does not restrict generality as long as a concrete value of the bound of the asymptotic regime, E_{as}, is not indicated. Evidently this condition does not play a part in calculation of $D(E, x)$ at $E < E_{as}$ due to the fast decrease with energy of the initial spectrum.

V. GENERAL CASE: FIRST APPROXIMATION

Consider the general case. Assuming analyticity of the ratio $D(E_v, x)/D_0(E_v)$ as a function of the variable v at the point $v = 0$, let us expand this function in a power series in v. This yields

$$D(E_v, x) = D_0(E_v) \left[1 + \sum_{n=1}^{\infty} v^n \hat{\partial}_n \right] [D(E, x)/D_0(E)] ,$$

where

$$\hat{\partial}_n \equiv \sum_{l=1}^{n} \binom{n-1}{l-1} \frac{E^l}{l!} \frac{\partial^l}{\partial E^l} .$$

Then, introducing the definitions

$$\Delta_n(E) = \int_0^1 \Phi(v, E_v) \eta(v, E) v^n dv , \quad n = 1, 2, \dots , \tag{5.1}$$

$$\mathcal{A}(E) = \int_0^1 [\Phi(v, E) - \eta(v, E)\Phi(v, E_v)] dv , \tag{5.2}$$

with $\eta(v, E) = (1 - v)^{-1} D_0(E_v)/D_0(E)$, we find

$$\langle D(E, x) \rangle = \left[\sum_{n=1}^{\infty} \Delta_n(E) D_0(E) \hat{\partial}_n D_0^{-1}(E) - \mathcal{A}(E) \right] D(E, x) . \tag{5.3}$$

Due to the fact that the functions $\Phi_k(v, E)$ depend rather slowly on E, and the initial muon spectrum $D_0(E)$ is close to a power-low one at high enough energies (vide supra), the ratio $D(E, x)/D_0(E)$ should be asymptotically a relatively slowly varying function of E. Thus the derivatives $D_0(E)\hat{\partial}_n D_0^{-1}(E)D(E, x)$ are small. It is obvious also that the integrals $\Delta_n(E)$ decrease with increasing n. Moreover, due to the specific v-dependence of the cross sections (see Sec. II B), $\Delta_1(E) \gg \Delta_n(E)$ at $n > 1$. These simple heuristic considerations allow us to use as a first approximation only two leading terms of the expansion (5.3). In this approximation Eq. (2.1) is merely a partial differential one,

$$\left[\frac{\partial}{\partial} - \beta_1(E) \frac{\partial}{\partial E} + \mathcal{R}(E) \right] D^{(1)}(E, x) = 0 , \tag{5.4}$$

where the following notations has been used:

$$\beta_1(E) = \beta_i(E) + \Delta_1(E)E \ , \qquad \mathcal{R}(E) = \mathcal{A}(E) - [g(E)+1]\Delta_1(E) - \beta_i'(E) \ ,$$

with $g(E) + 1 = -ED_0^{-1}(E)D_0'(E)$. We will assume subsequently that $g(E)$ is a positive definite and nondecreasing function. Clearly $g(E) = \gamma$ as $E \gg E_{as}$.

The solution to Eq. (5.4) can be expressed as

$$D^{(1)}(E,x) = D_0(\mathcal{E}_1(E,x)) \exp[-\mathcal{K}(E,x)] \equiv \mathcal{D}(E,x) \ , \tag{5.5}$$

where

$$\mathcal{K}(E,x) = \int_0^x \mathcal{R}(\mathcal{E}_1(E,x'))dx' = \int_E^{\mathcal{E}_1(E,x)} \frac{\mathcal{R}(E')dE'}{\beta_1(E')} \ . \tag{5.6}$$

The function $\mathcal{E}_1(E,x)$ can be obtained from the equation $\lambda_1(\mathcal{E}_1, E) = x$ (an analog of the energy-range relation), with

$$\lambda_1(E_1, E_2) = \int_{E_2}^{E_1} \frac{dE}{\beta_1(E)} \ .$$

The properties of the function $\mathcal{E}_1(E,x)$ are completely similar to the ones of above-mentioned function $\mathcal{E}(E,x)$, but the physical meaning of this quantity is not so obvious. Considering that the function $\beta_1(E)$ is an effective rate of the average energy loss (both continuous and discrete) for a given initial muon spectrum, the function $\mathcal{E}_1(E,x)$ can be interpreted as the effective (for the given $D_0(E)$) energy which a muon must have at the boundary of the medium in order to reach depth x having energy E *with a nonzero probability*. To refine this interpretation let us rewrite Eq. (5.5) in the form which is like the expression for the spectrum in the CL approximation (3.2):

$$\mathcal{D}(E,x) = D_0(\mathcal{E}_1(E,x))\frac{\beta_1(\mathcal{E}_1(E,x))}{\beta_1(E)}\mathcal{P}(\mathcal{E}_1(E,x), E) \ , \tag{5.7}$$

where

$$P(E_1, E_2) = \exp\left[-\int_{E_2}^{E_1} \frac{q(E')dE'}{\beta_1(E')}\right] \tag{5.8}$$

and

$$q(E) = \mathcal{R}(E) + \beta_1'(E) = \mathcal{A}(E) - g(E)\Delta_1(E) + \Delta_1'(E)E . \tag{5.9}$$

Evidently the function $q(E)$ reflects the effect of muon range straggling. It can be demonstrated that $q(E) > 0$ at least for high enough energies. Indeed, substituting Eqs. (5.1) and (5.2) into the right side of Eq. (5.9) yields

$$\begin{aligned} q(E) &= \int_0^1 \{\Phi(v, E) - [1 + g(E)v]\eta(v, E)\Phi(v, E_v)\}dv \\ &+ \int_0^1 [g(E_v) - g(E)]\Phi(v, E_v)\eta(v, E)vdv \\ &+ \int_0^1 E_v\frac{\partial\Phi(v, E_v)}{\partial E_v}\eta(v, E)vdv . \end{aligned} \tag{5.10}$$

The factor $[1 + g(E)v]\eta(v, E)$ does not exceed unit[4] and decreases fast (tends to zero) with increasing v, while the function $\Phi(v, E_v)$ depends on the second argument, E_v, only logarithmically. Thus the first integral in Eq. (5.10) is positive. The second integral is nonnegative on the assumption that $g(E)$ is an increasing (or constant) function. The third integral is small in comparison with the first one due to the factor $\eta(v, E)v$ in the integrand and (mainly) to the inequality

$$E_v\left|\frac{\partial\Phi(v, E_v)}{\partial E_v}\right| \ll \Phi(v, E_v) ,$$

which takes place even in the absence of the full screening [notice that $\gamma_Z(v, E_v) < 1$ at E above $\sim$1 TeV at any v]. Hence the last contribution cannot change the sign of the function $q(E)$.

---

[4]Because the derivative

$$\frac{\partial}{\partial v}\{[1 + g(E)v]\eta(v, E)\} = -g(E_v)\left\{1 - \frac{g(E)}{g(E_v)} + [1 + g(E)]\frac{v}{1 - v}\right\}\eta(v, E)$$

Thus the function $q(E)$ can be interpreted as an effective absorption coefficient dependent upon the radiative and photonuclear energy losses, and the function $\mathcal{P}(\mathcal{E}_1(E, x), E)$ should be treated as the probability for a muon with energy $\mathcal{E}_1(E, x)$ at the surface to reach depth x with energy E.

Simple examination shows that $\Delta_1(E) < b(E)$. Therefore, $\mathcal{E}_1(E, x) < \mathcal{E}(E, x)$ for all values of E and x. Moreover, the remainder $\mathcal{E}(E, x) - \mathcal{E}_1(E, x)$ increases fast with depth since

$$\frac{\partial}{\partial x}[\mathcal{E}(E, x) - \mathcal{E}_1(E, x)] = \beta(\mathcal{E}(E, x)) - \beta_1(\mathcal{E}_1(E, x))$$

$$= \beta_i(\mathcal{E}) - \beta_i(\mathcal{E}_1) + b(\mathcal{E})\mathcal{E} - \Delta_1(\mathcal{E}_1)\mathcal{E}_1 > 0 \ ,$$

and we have taken into account that $\beta_i(E)$ is a nondecreasing function of E after a broad minimum at $p \approx 300$ MeV/c, almost independently of the medium [30]. It is obvious also that the remainder $\mathcal{E}(E, x) - \mathcal{E}_1(E, x)$ increases when the slope of the initial muon spectrum grows. The decrease of the minimal muon energy at the surface, necessary in order that a muon can reach a given depth with a given energy, is an evident reflection of the discreteness of radiative and photonuclear muon energy losses. The function $\mathcal{E}_1(E, x)$ is a useful approximation to estimate this minimal energy within the scope of the approximate solution (5.5).

From Eqs. (5.7-5.9) the following expression for the integral spectrum in the first approximation can be obtained:

$$I^{(1)}(E, x) = \int_{\mathcal{E}_1(E, x)}^{\infty} D_0(E')\mathcal{P}(E', \mathcal{E}_1(E', -x))dE' \ ,$$

which is an evident generalization of Eq. (3.3) obtained in the CL approximation. It is obvious that $I^{(1)}(E, x) < I_0(\mathcal{E}_1(E, x))$. In the realistic case, when $q(E)$ is a function slowly varying with energy we find

$$I^{(1)}(E, x) \simeq I_0(\mathcal{E}_1(E, x))e^{-\overline{q}x} \, ,$$

where $\overline{q}$ is the average of $q(E)$.

Consider now the approximate solution (5.5) in the SPS model. It is clear that all moments

$$\Delta_n = \int_0^1 \Phi(v)(1 - v)^\gamma v^n dv \equiv \Delta_n^\gamma$$

(in particular, $\Delta_1 = \Delta_1^\gamma \equiv \varrho_\gamma$) and the parameter $\mathcal{A} = b_\gamma$ are constant in this case. So the effective absorption coefficient $q = b_\gamma - \gamma\varrho_\gamma \equiv q_\gamma$ is a positive constant, such that

$$\frac{1}{2}\gamma(\gamma + 1)\Delta_2^\gamma < q_\gamma < \gamma^2\Delta_2^0 \, .$$

One can easily show that

$$\mathcal{E}_1^{\text{SPS}}(E, x) = E[(1 + \xi)e^{\varrho_\gamma x} - \xi] \quad \text{and} \quad \mathcal{P}(\mathcal{E}_1^{\text{SPS}}(E, x), E) = e^{-q_\gamma x} \, .$$

By this means the differential and the integral spectra can be written as

$$D_{\text{SPS}}^{(1)}(E, x) = D_0^\gamma(E)e^{-b_\gamma x}\left[1 + \xi(1 - e^{-\varrho_\gamma x})\right]^{-(\gamma+1)} \, , \tag{5.11}$$

$$I_{\text{SPS}}^{(1)}(E, x) = I_0^\gamma(E)e^{-b_\gamma x}\left[1 + \xi(1 - e^{-\varrho_\gamma x})\right]^{-\gamma} \, , \tag{5.12}$$

and DIR becomes

$$J_{\text{SPS}}^{(1)}(x) = I_0^\gamma(W_\gamma(e^{\varrho_\gamma x} - 1))e^{-q_\gamma x} \, ,$$

for any $E_t \ll W_\gamma(1 - e^{-\varrho_\gamma x})$, where $W_\gamma = a/\varrho_\gamma$. Thus, at large depths,

$$J_{\text{SPS}}^{(1)}(x)/\overline{J}_{\text{SPS}}(x) \simeq (\varrho_\gamma/b)^\gamma e^{(\gamma b - b_\gamma)x}$$

As might be expected, the first two terms of the exact $1/E$-expansion (4.5) coincide with

(5.11) has the correct behavior at high energies at least when $\zeta \leq 1$ (see (4.8)). It should be noted also that the approximation (5.11) is self-consistent. Indeed, one can show that

$$\hat{\partial}_n \left[\frac{D_{\text{SPS}}^{(1)}(E,x)}{D_0^\gamma(E)} \right] = \frac{(\gamma+1)_n}{n!} Z^n(E,x) \left[\frac{D_{\text{SPS}}^{(1)}(E,x)}{D_0^\gamma(E)} \right] ,$$

where

$$Z(E,x) = \frac{\xi f_1(x)}{1 + \xi f_1(x)} = \frac{\xi(1 - e^{-\varrho_\gamma x})}{1 + \xi(1 - e^{-\varrho_\gamma x})} .$$

Considering that $Z(E,x) < 1$ at any E and x the series in the right side of Eq (5.3) is always uniformly convergent and one may actually cut it off after the 1st term if

$$\frac{(\gamma+2)}{2} \frac{\Delta_2^\gamma}{\Delta_1^\gamma} Z(E,x) \ll 1 ,$$

and that is certainly admissible when $\xi f_1(x) \ll 1$. This is supporting the approximation (5.5) as a suitable ansatz.

VI. GENERAL CASE: ITERATION SCHEME

Let us now represent the solution to Eq. (2.1) by the following form

$$D(E,x) = D^{(1)}(E,x)[1 + \delta(E,x)] , \tag{6.1}$$

where $\delta(E,x)$ is an unknown function ("relative correction"). To derive the equation for $\delta(E,x)$ it is convenient at first to rewrite Eq. (5.4) as

$$\frac{\partial}{\partial x}\mathcal{D}(E,x) - \frac{\partial}{\partial E}[\beta_i(E)\mathcal{D}(E,x)] = [\Delta_1(E)\omega(E,x) - \mathcal{A}(E)]\mathcal{D}(E,x) , \tag{6.2}$$

where

$$\omega(E,x) = \frac{E[h(E) - h(\mathcal{E}_1(E,x))]}{\beta_1(E)} , \tag{6.3}$$

$$h(E) = \mathcal{R}(E) + \frac{[g(E)+1]\beta_1(E)}{E} = \mathcal{A}(E) + \frac{[g(E)+1]\beta_i(E)}{E} - \beta_i'(E) . \tag{6.4}$$

In order to derive Eq. (6.2) we have used Eqs. (5.5) and (5.6). Direct substitution of Eq. (6.1) into Eq. (2.1), in view of Eq. (6.2), then gives

$$\hat{\mathcal{L}}_i \delta(E, x) = \int_0^1 \Phi(v, E_v) \left\{ \Omega(E, x; v)[1 + \delta(E_v, x)] \right.$$
$$\left. - [1 + \omega(E, x)v][1 + \delta(E, x)] \right\} \eta(v, E) dv , \tag{6.5}$$

where the differential operator

$$\hat{\mathcal{L}}_i = \frac{\partial}{\partial x} - \beta_i(E)\frac{\partial}{\partial E}$$

was introduced, and we have defined

$$\Omega(E, x; v) = \frac{D_0(E)}{D_0(E_v)} \frac{D_0(\mathcal{E}_1(E_v, x))}{D_0(\mathcal{E}_1(E, x))} \exp[\mathcal{K}(E, x) - \mathcal{K}(E_v, x)] . \tag{6.6}$$

Clearly $\delta(E, 0) = 0$. We shall seek the solution to Eq. (6.5) using a procedure of successive approximations.

Let us note initially that the function $\delta(E, x)$ follows a $c_2(x)/E^2$-dependence as $E \gg E_{as}$, where $c_2(x)$ is independent of energy. It is a straight corollary of the coincidence of the first two terms in the $1/E$-expansions for the approximate solution (5.5) and the exact SPS solution (4.5). Therefore

$$\Theta(E, x; v) = \delta(E_v, x) - (1 - v)^2 \delta(E, x) \propto (1 - v)^2 v/E^3$$

as $E \gg E_{as}$, i.e. the function $\Theta(E, x; v)$ is small in absolute value by comparison with $\delta(E, x)$. We assume that at all energies the term with the factor $\Theta(E, x; v)$ can be neglected in the integrand of the right side of Eq. (6.5) as a first approximation. Thus the equation for the correction function in *second* approximation becomes

where

$$R_l(E, x) = \int_0^1 \Phi(v, E_v) \left\{ \Omega(E, x; v)(1 - v)^l - [1 + \omega(E, x)v] \right\} \eta(v, E) dv \ . \tag{6.8}$$

Solving Eq. (6.7) yields

$$\delta^{(2)}(E, x) = \int_0^x \exp\left[\int_{x'}^x R_2(\mathcal{E}_i(E, x - x''), x'') dx'' \right] R_0(\mathcal{E}_i(E, x - x'), x') dx'$$

$$\equiv \int_E^{\mathcal{E}_i(E,x)} \exp\left[\int_{E'}^{\mathcal{E}_i(E,x)} \frac{R_2(E'', x - \lambda_i(E, E''))}{\beta_i(E'')} dE'' \right] \frac{R_0(E', x - \lambda_i(E, E'))}{\beta_i(E')} dE' \ , \tag{6.9}$$

where $\mathcal{E}_i(E, x)$ is the only root of the equation $\lambda_i(\mathcal{E}_i, E) = x$, and

$$\lambda_i(E_1, E_2) = \int_{E_2}^{E_1} \frac{dE}{\beta_i(E)}$$

is the ionization range of a muon with initial energy E_1 and final energy E_2 (hence $\mathcal{E}_i(E, x) - E$ is simply the energy lost due to ionization).

Let us consider one evident consequence of Eq. (6.9). Clearly $R_2(E, x) < R_0(E, x)$ for all values of the arguments. Substituting this inequality into Eq. (6.9), and integrating over E' then gives

$$\exp[K_2(E, x)] \le 1 + \delta^{(2)}(E, x) \le \exp[K_0(E, x)] \ , \tag{6.10}$$

with

$$K_l(E, x) = \int_0^x R_l(\mathcal{E}_i(E, x - x'), x') dx' \ .$$

The exponential factors in (6.10) can be treated as the lower and upper limits for the correction to the "survival probability" $\mathcal{P}(\mathcal{E}_1(E, x), E)$ so long as

$$\int_0^x [q(\mathcal{E}_1(E, x - x')) - R_0(\mathcal{E}_i(E, x - x'), x')] dx' > 0 \ .$$

In order to build an equation for calculation of the correction function in the l-th ap-

$c_3(x)/E^3$ with an E-independent function $c_3(x)$, as it can be easily verified using Eqs. (6.5) and (6.7). Therefore, in the next approximation we may put approximately

$$\delta(E_v, x) - \delta^{(2)}(E_v, x) \simeq (1 - v)^3 [\delta(E, x) - \delta^{(2)}(E, x)] .$$

Repeating the consideration we find by induction that $\delta(E, x) - \delta^{(l)}(E, x) \to c_l(x)/E^l$ as $E \gg E_{\mathrm{as}}$, and thus we put

$$\delta(E_v, x) - \delta^{(l)}(E_v, x) \simeq (1 - v)^{l+1} [\delta(E, x) - \delta^{(l)}(E, x)] . \tag{6.11}$$

Let us define

$$\Theta_l(E, x) = \delta^{(l)}(E, x) - \delta^{(l-1)}(E, x) , \quad l \geq 2 , \tag{6.12}$$

with $\delta^{(1)}(E, x) \equiv 0$ by definition. From Eq. (6.5), using Eq. (6.12) and Eq. (6.11) we obtain the recursion chain of equations for the functions $\Theta_l(E, x)$:

$$[\hat{\mathcal{L}}_i - R_l(E, x)]\Theta_l(E, x) = \Re_{l-1}(E, x) , \quad l \geq 3 , \tag{6.13}$$

where

$$\Re_l(E, x) = \int_0^1 \Phi(v, E_v)\Omega(E, x; v)[\Theta_l(E_v, x) - (1 - v)^l \Theta_l(E, x)]\eta(v, E)dv . \tag{6.14}$$

The solution to Eq. (6.13) is given by

$$\Theta_l(E, x) = \int_0^x \exp\left[\int_{x'}^x R_l(\mathcal{E}_i(E, x - x''), x'')dx''\right] \Re_{l-1}(\mathcal{E}_i(E, x - x'), x')dx'$$

$$\equiv \int_E^{\mathcal{E}_i(E,x)} \exp\left[\int_{E'}^{\mathcal{E}_i(E,x)} \frac{R_l(E'', x - \lambda_i(E, E''))}{\beta_i(E'')}dE''\right] \frac{\Re_{l-1}(E', x - \lambda_i(E, E'))}{\beta_i(E')}dE' . \tag{6.15}$$

To verify the convergency of the iteration procedure consider firstly the behavior of the function $R_l(E, x)$ at $l \gg 1$. Due to the factor $(1 - v)^l$ in the first term of the integrand of Eq. (6.8) and the properties of the macroscopic cross sections (see Sec. II B), only the region

of small values of v is important in this case. So at $l \gg 1$ the function $\Omega(E, x; v)$ can be estimated as

$$\Omega(E, x; v) \simeq \Omega(E, x; 0) + v \left[\frac{\partial \Omega(E, x; v)}{\partial v} \right]_{v=0}$$

or, considering the definitions (6.3), (6.4), and (6.6),

$$\Omega(E, x; v) \simeq 1 + \omega(E, x)v \sim 1 . \tag{6.16}$$

Thus

$$R_l(E, x) \to - \int_0^1 \Phi(v, E_v)[1 + \omega(E, x)v][1 - (1 - v)^l]\eta(v, E)dv$$

at $l \gg 1$ and, therefore, $R_l(E, x) < 0$ and $|R_l(E, x)|$ increases indefinitely with l. Clearly the exponential factor

$$\exp \left[\int_{x'}^{x} R_l(\mathcal{E}_i(E, x - x''), x'')dx'' \right]$$

in the integrand of Eq. (6.15) diminishes fast with increasing l.

On the other hand, in view of the fact that $\Theta_l(E, x) \propto E^{-l}$ at energies high enough, we can write $\Theta_l(E_v, x) - (1 - v)^l \Theta_l(E, x) = v(1 - v)^l F_l(E, x; v)$, where $F_l(E, x; v)$ is a function which can be estimated at $v \ll 1$ by

$$F_l(E, x; 0) = E \frac{\partial \Theta_l(E, x)}{\partial E} + l\Theta_l(E, x) \propto E^{-(l+1)} .$$

Hence, using Eq. (6.16), the integral (6.14) can be estimated at $l \gg 1$ as

$$\Re_l(E, x) \simeq F_l(E, x; 0) \int_0^1 \Phi(v, E_v)[1 + \omega(E, x)v]\eta(v, E)(1 - v)^l v dv .$$

Thus $\Re_l(E, x)$ and $\Re_l(\mathcal{E}_i(E, x - x'), x')$ are positive and decrease when l increases, if the function $F_l(E, x; 0)$ is bounded in magnitude as $l \to \infty$ or even if $|F_l(E, x; 0)|$ increases with l not too fast. This can be verified by induction.

The foregoing proves that $\Theta_l(E, x) \to 0$ as $l \to \infty$ for any depths at least for high enough energies. Due to the correct asymptotic behavior of the functions $\Theta_l(E, x)$ at all values of l this indicates that the iteration procedure converges, that is

$$\delta(E, x) = \lim_{l \to \infty} \delta^{(l)}(E, x) .$$

A more cumbersome analysis and numerical verifications demonstrate that this statements is true *at all energies* under quite general assumptions on the energy dependence of the rate of the continuous energy loss, the macroscopic cross sections, and the initial spectrum; specifically if the functions $\beta_i(E)$ and $\Phi(v, E)$ increase monotonically and sufficiently slowly, while $D_0(E)$ decreases with energy so that $g(E)$ is a slightly varying function of energy. As it follows from numerical estimations, the convergency rate is very good, and usually only a few iterations are needed to reach an accuracy of the order of 1% at depths up to ~ 20 km of water equivalent for muon energies above ~ 1 GeV.

By way of illustration let us again direct our attention to the SPS model and consider the second approximation correction function $\delta_{\mathrm{SPS}}^{(2)}(E, x)$. Under condition (4.8) we can write the exact expression for the correction function, using definition (6.1) and Eqs. (4.5) and (5.11):

$$\delta_{\mathrm{SPS}}(E, x) = \sum_{n=2}^{\infty} \frac{(\gamma + 1)_n}{n!} \frac{f_n(x) - [f_1(x)]^n}{[1 + \xi f_1(x)]^{\gamma+1}} (-\xi)^n . \tag{6.17}$$

Therefore at $\zeta \le 1$ the leading asymptotic term has the form

$$c_\gamma \epsilon \left[a_0 + a_1 e^{-\varrho_\gamma x} + a_1 e^{-2\varrho_\gamma x} - (a_0 + a_1 + a_2)e^{-(2-\epsilon)\varrho_\gamma x} \right] \xi^2 , \tag{6.18}$$

where $c_\gamma = (\gamma + 1)(\gamma + 2)/2$, $a_0 = 1/(2 - \epsilon)$, $a_1 = -2/(1 - \epsilon)$, $a_2 = -1/\epsilon$, and $\epsilon = \Delta_2^\gamma/\Delta_1^\gamma$. The expression for $R_l(E, x)$ in the SPS is given by

$$R_l(E, x) = R_l^\gamma(Z(E, x)) \tag{6.19}$$

$$R_l^\gamma(z) = \int_0^1 \Phi(v) \left\{ \frac{(1-v)^j}{(1-zv)^{\gamma+1}} - [1 + (\gamma+1)zv] \right\} (1-v)^\gamma dv .$$

In particular, at $z \ll 1$ we have

$$R_0^\gamma(z) \simeq c_\gamma \Delta_2^\gamma z^2 , \quad R_2^\gamma(z) \simeq -(2-\epsilon)\Delta_1^\gamma . \tag{6.20}$$

One can easily show, using Eqs. (6.19) and (6.20) that the asymptotic behavior of the exact correction function (6.18) is reproduced, as was to be expected, by the correction function in the second approximation (see Eq. (6.9)). It is important that the approximate correction $\delta_{\mathrm{SPS}}^{(2)}(E, x)$ is definite and bounded at all values of E and x, in contrast with the exact expression (6.17) given by a series which converges only under condition (4.8). This can be seen from the constraint (6.10). Indeed, a little manipulation, and taking into account that

$$\partial Z(E, x)/\partial E \leq 0 \text{ and } \partial Z(E, x)/\partial x \geq 0$$

yields

$$\begin{aligned}
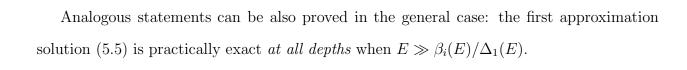K_0(E, x) &= \sum_{n=2}^\infty \frac{(\gamma+1)_n}{n!} \Delta_n^\gamma \int_0^x Z^n(E + ax', x - x') dx' \\
&\leq x R_0^\gamma(Z(E, x)) Z(E + ax, x)/Z(E, x) \\
&\leq x R_0^\gamma \left(\frac{\xi}{1+\xi} \right) \frac{1+\xi}{1+\xi+\zeta} \equiv K_0^{\max}(E, x) .
\end{aligned}$$

Hence

$$0 \leq \delta_{\mathrm{SPS}}^{(2)}(E, x) \leq \exp[K_0^{\max}(E, x)] - 1 .$$

and, therefore, the survival probability calculated in the second approximation, does not exceed the factor $\exp\{-[q_\gamma x - K_0^{\max}(E, x)]\}$. It is obvious that the function $K_0^{\max}(E, x)$ and, therefore, the correction $\delta_{\mathrm{SPS}}^{(2)}(E, x)$ are both bounded at all finite values of E and x. A significant consequence resides in the fact that $\delta_{\mathrm{SPS}}^{(2)} \ll 1$ at $\xi \ll 1$ for any x since $K_0^{\max} \simeq c_\gamma \epsilon \xi \zeta/(1+\zeta) \ll 1$. In other words, the first approximation solution (5.11) is correct

Analogous statements can be also proved in the general case: the first approximation solution (5.5) is practically exact *at all depths* when $E \gg \beta_i(E)/\Delta_1(E)$.

VII. SUMMARY AND OUTLOOK

The method described enables us to calculate with a controlled precision the differential energy spectra of CR muons after propagation through thick layers of matter. It is appropriate for any depths when muon energies are high enough and provides a way for including the real (non-power-law) initial muon spectrum and the energy variation of the continuous and discrete muon energy losses, with only the formal (but rather natural) requirement for the asymptotic behavior of the initial spectrum and the cross sections. A computer implementation of the method is fully straightforward and the required CPU time is small. This enables to use the method in on-line processing for underground/underwater experiments. It is important that the useful notion of the minimal muon energy at the boundary $\mathcal{E}(E, x)$, which has a physical sense when the range straggling is negligible (i.e. for small depths), has an analog ($\mathcal{E}_1(E, x)$) in the case of *arbitrary large fluctuations* (i.e. for arbitrary depths) when the first approximation solution $\mathcal{D}(E, x)$ is available (high muon energies, specifically $E >$ a few TeV).

We intend in the near future to give a detailed numerical illustration of the convergency of the iteration procedure besides calculating results on the CR muon energy spectra (differential and integral) and DIR at large depths for different types of rocks and water with varying charm production models *etc.* In recent years a rather representative array of data on DIR in rock and (to a lesser extent) in water has been accumulated by many experiments. One should systematize all these data and compare them against theoretical predictions.

We also look forward to analyze future underground and underwater experiments to possibly throw light on the prompt muon problem and to derive information about super-high-energy muon interactions with nuclei, primarily about the rather poorly studied photonuclear interaction.

ACKNOWLEDGMENTS

APPENDIX:

van Ginneken's parameterization of the v-distributions
for radiative processes

We give here a parameterization of the normalized cross sections $f_k(v, E) = \sigma_k^{-1} d\sigma_k/dv$ (where $d\sigma_k/dv \equiv d\sigma_k^{Z,A}/dv$) for pair production and bremsstrahlung as a function of the fractional energy loss v suggested in Ref. [32]. The formulas presented below are valid at muon energies from ~ 100 GeV up to 30 TeV and enable one to estimate the comparative probabilities of "soft" ($v \ll 1$) and "hard" ($v \sim v_{\max}^k \sim 1$) losses in the radiative processes.

$\bullet$ Direct pair production off both nuclear and electron targets $\bullet$

$$
\begin{aligned}
f_p^{(n,e)}(v, E) \quad &\propto \quad const\,, \quad 5m_e/E < v < 25m_e/E\,, \\
&\propto \quad v^{-1}\,, \quad 25m_e/E < v < v_1 \ (\text{if } E > 25m_e/v_1)\,, \\
&\propto \quad v^{-2}\,, \quad v_1 < v < v_2 \ (\text{if } E > 25m_e/v_2)\,, \\
&\propto \quad v^{-3}\,, \quad v_2 < v < 1\,,
\end{aligned}
$$

where $v_1 = 0.002$ and $v_2 = 0.02$. For very small v (up to the kinematic limit $v_{\min}^p(E) = 4m_e/E$) $d\sigma_p(v, E)/dv \propto \sigma_0(vE) \ln(1/v)v^{-1}$, where $\sigma_0(\nu)$ is the total cross section for pair production by a photon of energy ν (Kel'ner's approximation). Thus $d\sigma_p/dv$ follows roughly

Below $\nu \approx 5m_e$, $\sigma_0(\nu)$ remains small $(< 0.05\sigma_0(\infty))$, and it increases roughly linearly until $\nu \approx 25m_e$, where $\sigma_0(\nu) \approx 0.5\sigma_0(\infty)$.

• Bremsstrahlung contribution off a nuclear target •

$$
\begin{aligned}
f_b^{(n)}(v, E) \; &\propto \; v^{-1}, & v_{\min}^b(E) < \; &v \; < 0.03, \\
&\propto \; v^{C_n(E)}, & 0.03 < \; &v \; < v_b^n(E), \\
&\propto \; (1-v)^{C_n'(E)}, & v_b^n(E) < \; &v \; < 1,
\end{aligned}
$$

where $v_{\min}^b(E) = 0.001/E$, $v_b^n(E) = (1 + 4.5/\sqrt{E})^{-1}$, $C_n(E) = 1.39 - 0.024 \ln E$, and $C_n'(E) = 1.32 - 0.12 \ln E$. Only for values of v above 0.995 the parameterization is not very reliable, but it is not important in practice.

• Bremsstrahlung contribution off atomic electrons •

$$
\begin{aligned}
f_b^{(e)}(v, E) \; &\propto \; v^{-0.95}, & v_{\min}^b(E) < \; &v \; < 0.05, \\
&\propto \; v^{C_e(E)}, & 0.05 < \; &v \; < v_b^e(E), \\
&\propto \; (v_{\max}^b(E) - v)^{1/2}, & v_b^e(E) < \; &v \; < v_{\max}^b(E),
\end{aligned}
$$

where $v_{\max}^b(E) = (1 + 10.92/E)^{-1}$, $v_b^e(E) = v_b^n(E)v_{\max}^b(E)$, and $C_e(E) = 1.50 - 0.03 \ln E$.

The proportionality factors which was omitted in the above parameterization can be obtained by continuity and normalization. A slight Z-dependence in the v-distributions has been ignored. The energy E and the electron mass m_e have been expressed in GeV.

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